Ph.D. Course on
"Vorticity, Vortical Flows and Vortex-Induced Vibrations"
26-30 August 2019, Technical University of Denmark

Lecture 4

Long- and short-wave instabilities

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cnrs

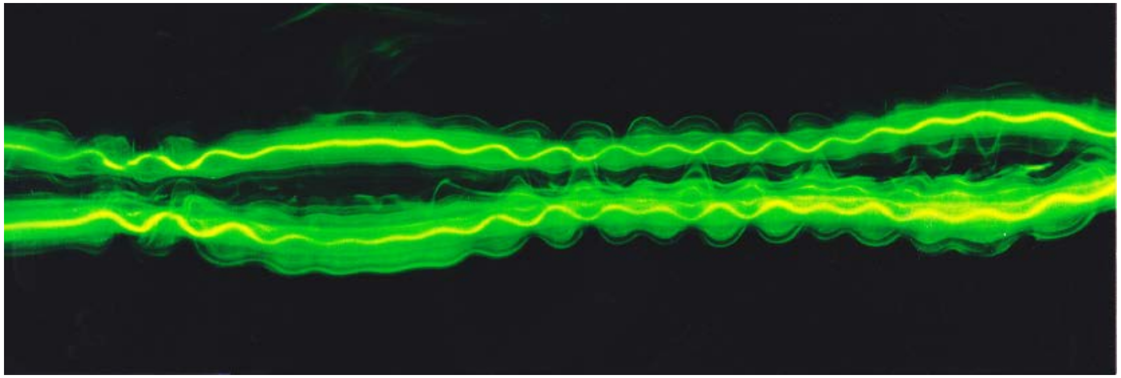
Long-wave instabilities of a vortex / system

- displacement / deformation of the vortex shape
- description by a filament approach

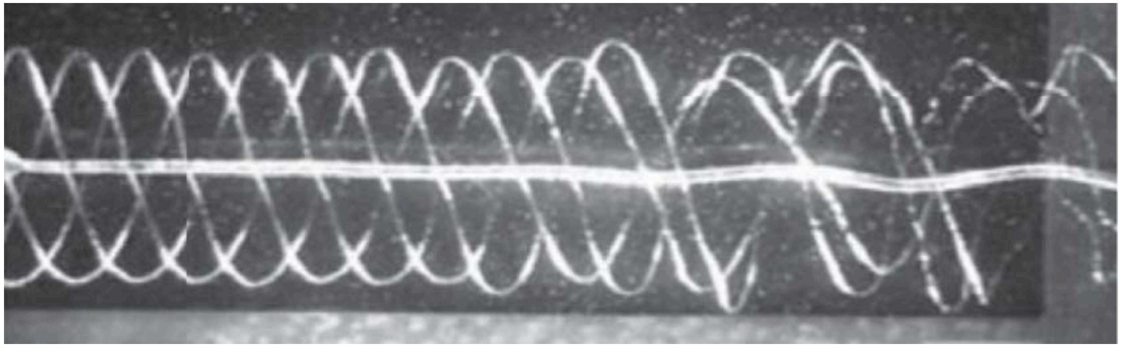
Short-wave instabilities

- internal deformations of the vortex core
- interaction of vortex Kelvin modes

Vortex pairs



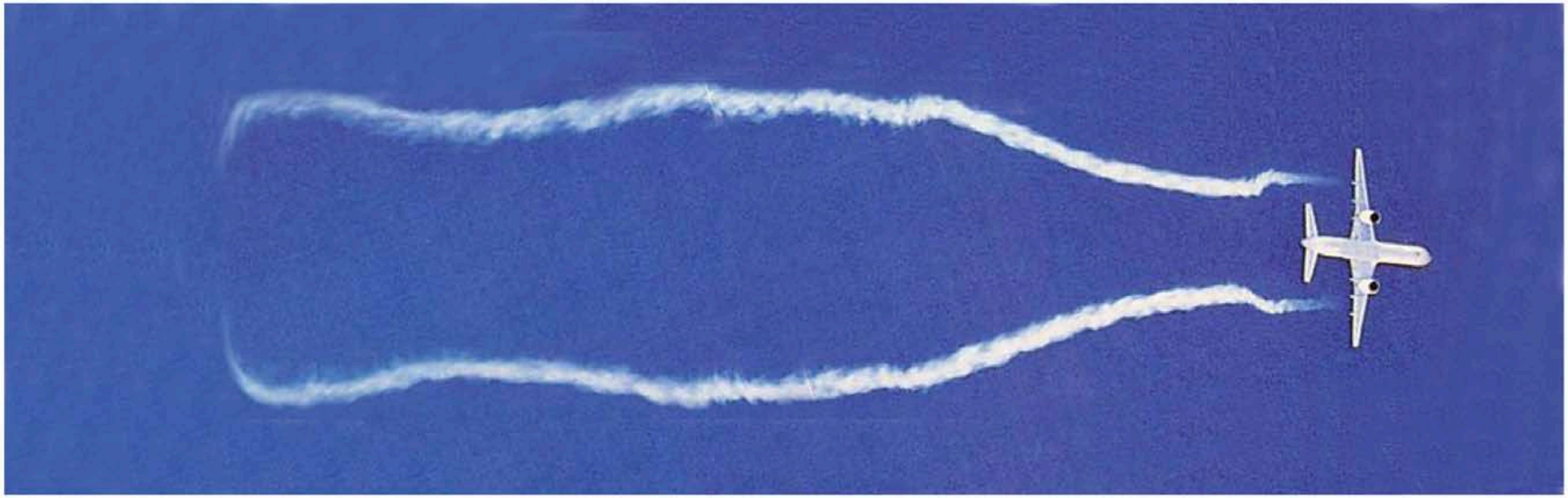
Helical vortices

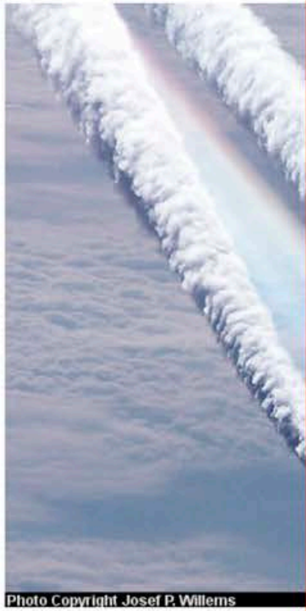


Long-wave instability
Vortex pairs

Crow instability

Visualisations of aircraft trailing wakes

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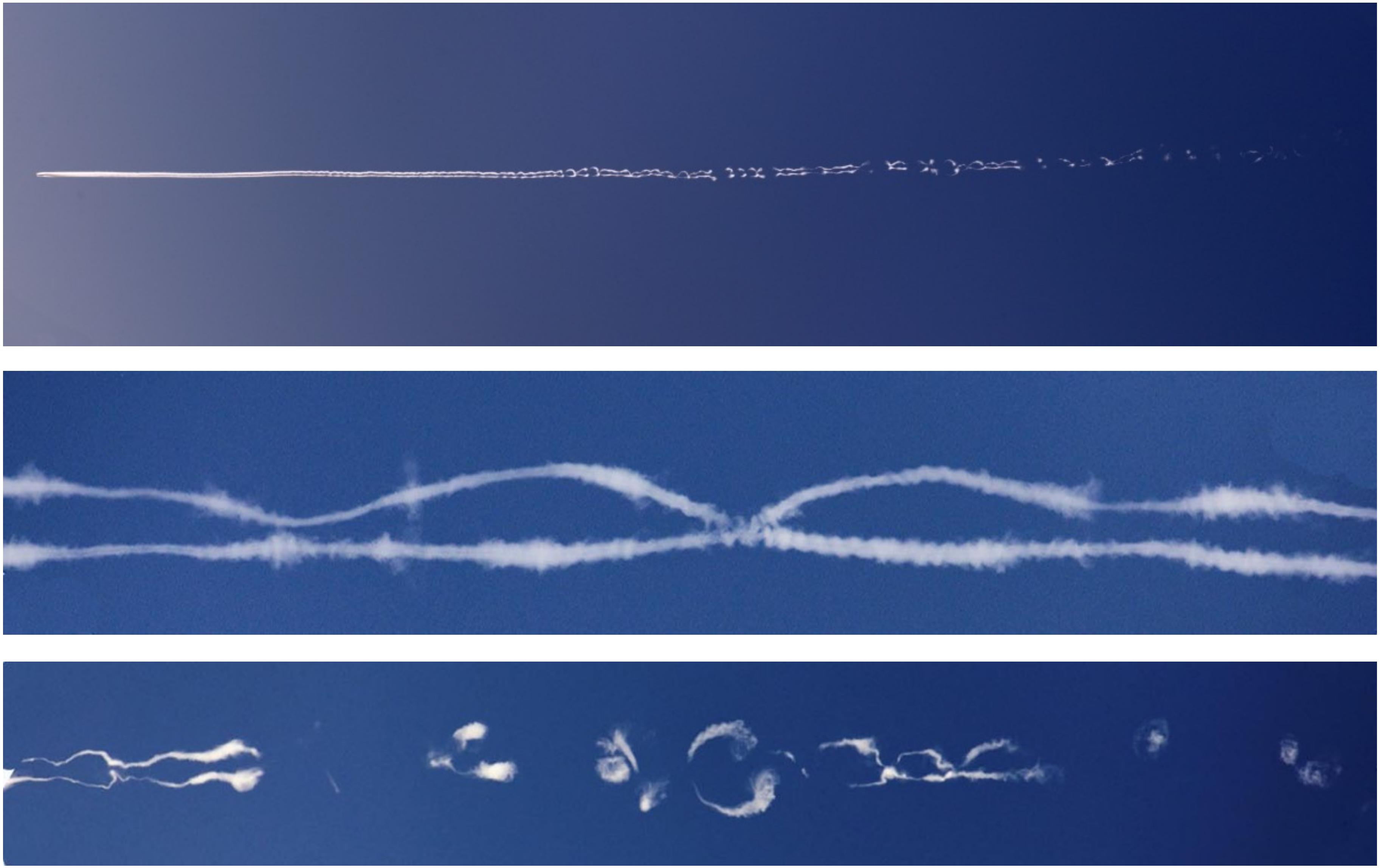
Sempre
always
Coca-Cola
SEMPRE A BORDO. SEMPRE REFRESCANTE.


van Dijck Fotografie

Long-wavelength Crow instability



Crow instability

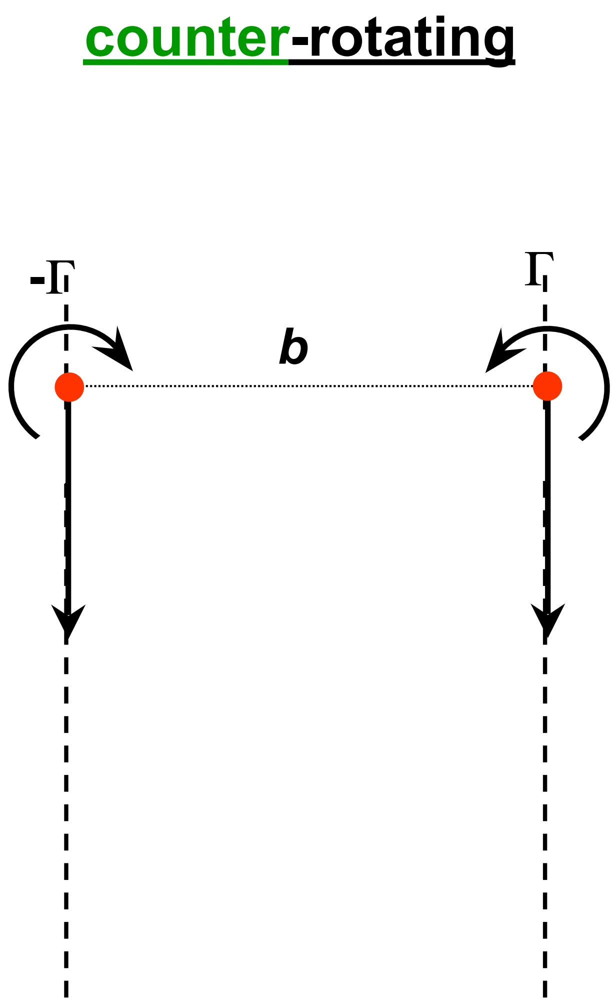
Only natural way for aircraft wake decay in a calm atmosphere

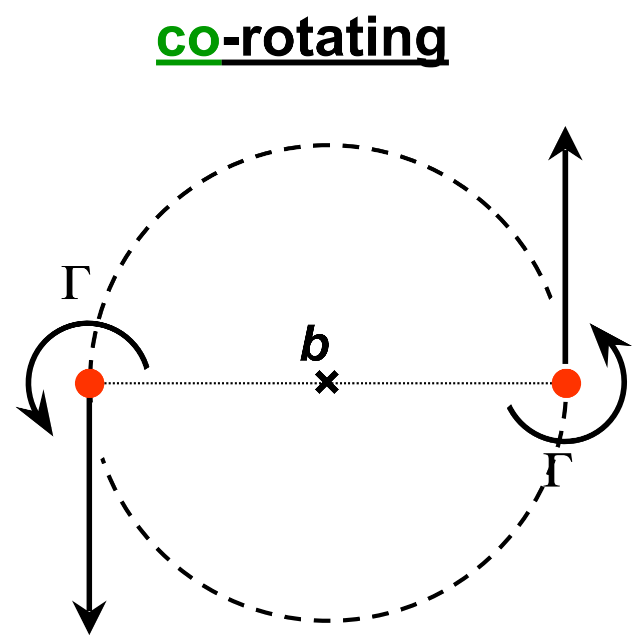
NASA Langley Research Center
Wake Vortex Experiments

NASA Langley Research Center
Wake Vortex Experiments

Dynamics of two point vortices

- same circulation -

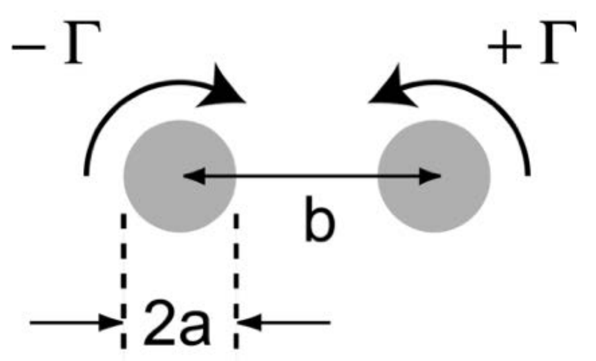




mutually induced velocity:
$|V| = \Gamma / 2\pi b$

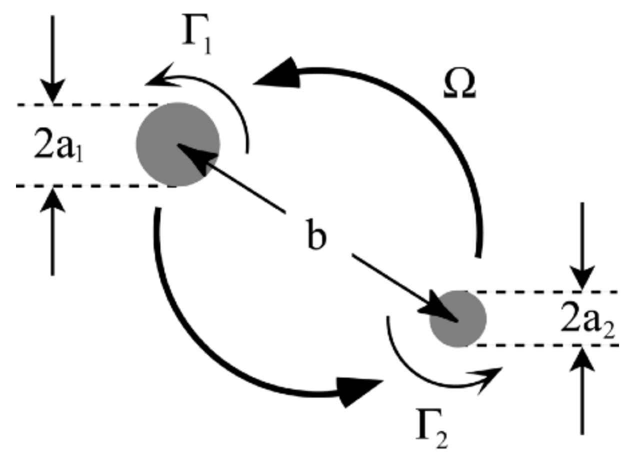
Vortex pair parameters

counter-rotating



- vortex separation ***b***
- descent speed $V = \Gamma / 2\pi b$
- non-dim. time $t^* = t V / b$

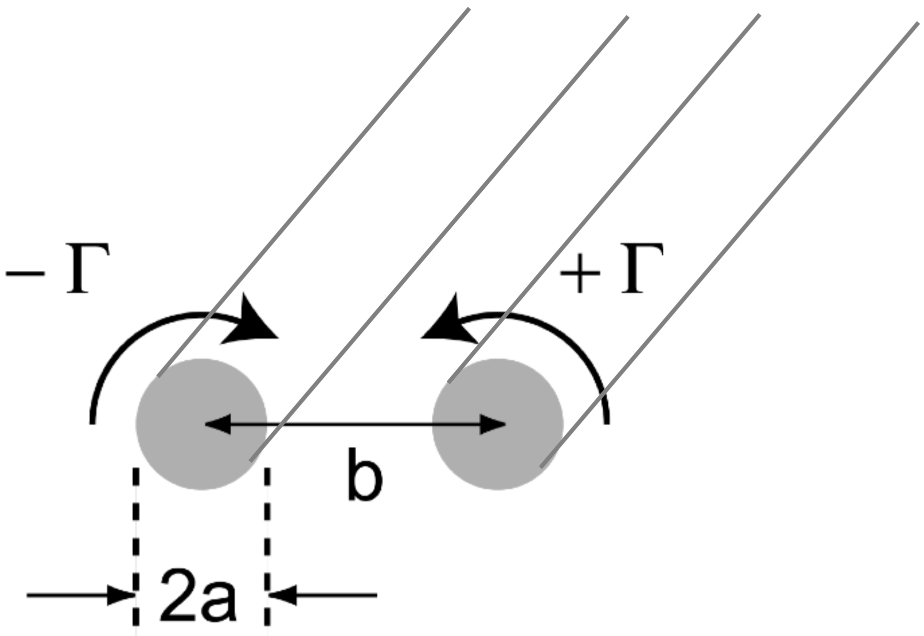
co-rotating

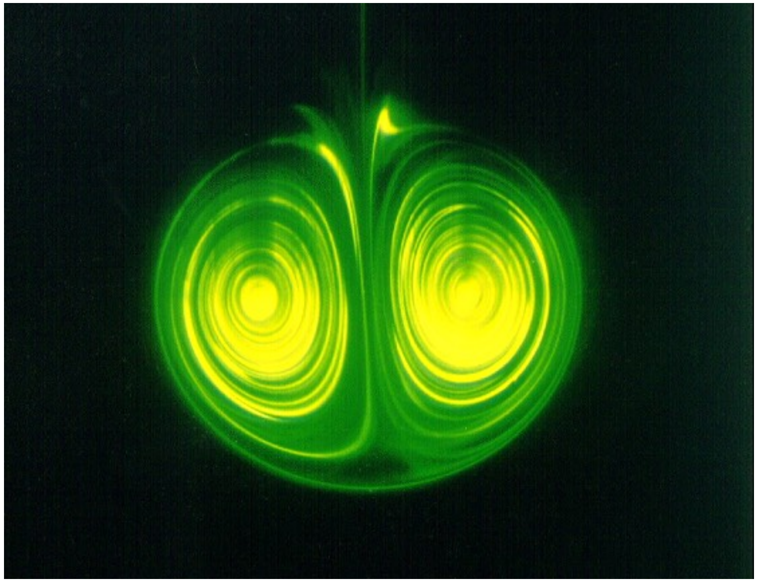


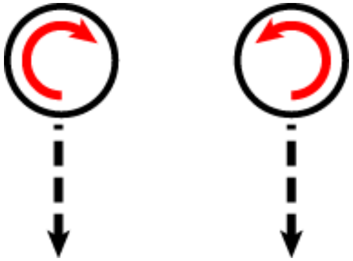
- vortex separation ***b***
- angular freq. $\Omega = (\Gamma_1 + \Gamma_2) / 2\pi b^2$
- non-dim. time $t^* = t\, \Omega / 2\pi$

rate of strain induced by one vortex on the other: $\varepsilon = \Gamma / 2\pi b^2$

Counter-rotating vortex pair







Basic state:
vertical translation
with speed $V = \Gamma / 2\pi b$

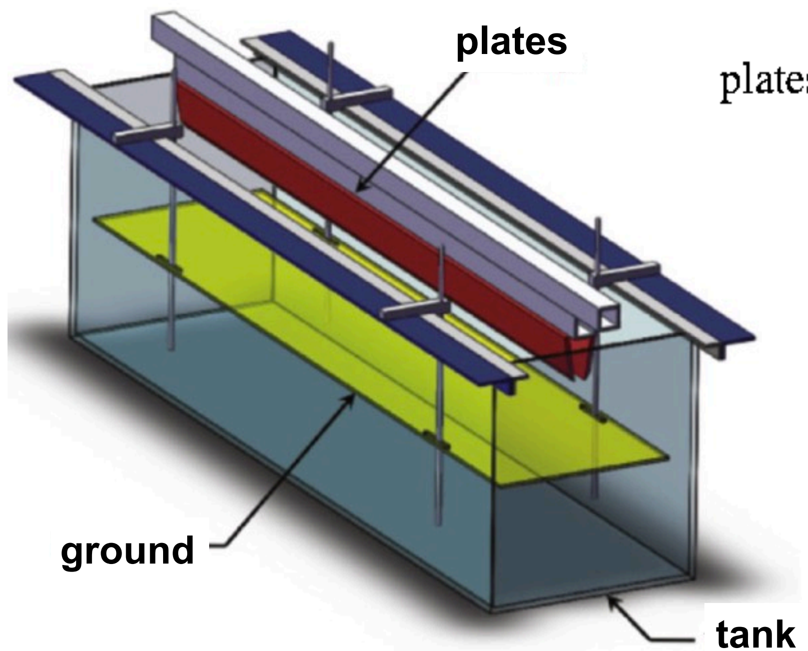
Counter-rotating vortex pairs

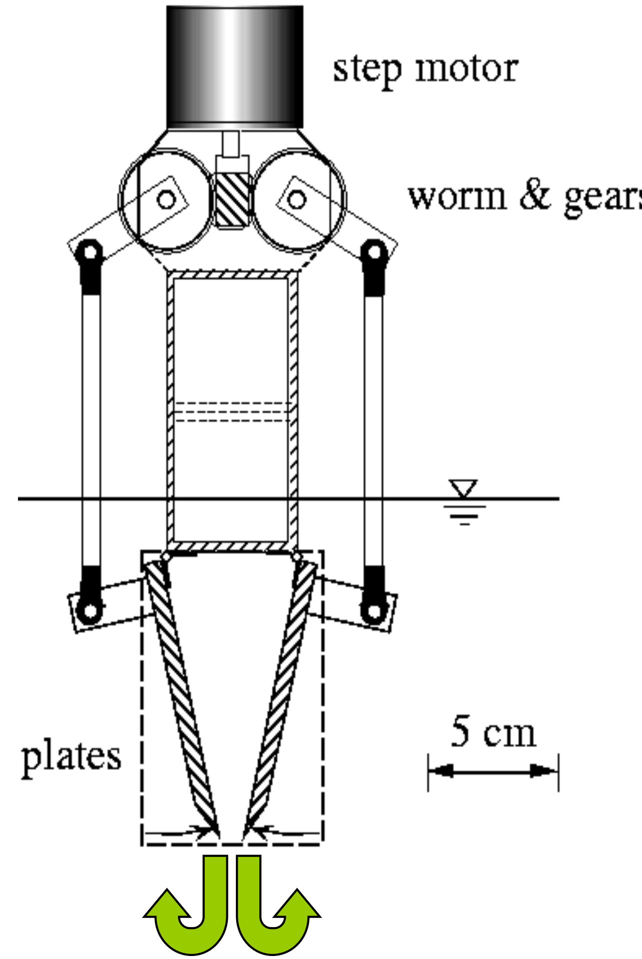
Experimental set-up

- water tank (180 × 45 × 60 cm³)
- starting vortices generated by impulsively rotated plates
- computer-controlled plate motion

methods:

- visualisation
- image analysis
- Particle Image Velocimetry (PIV)

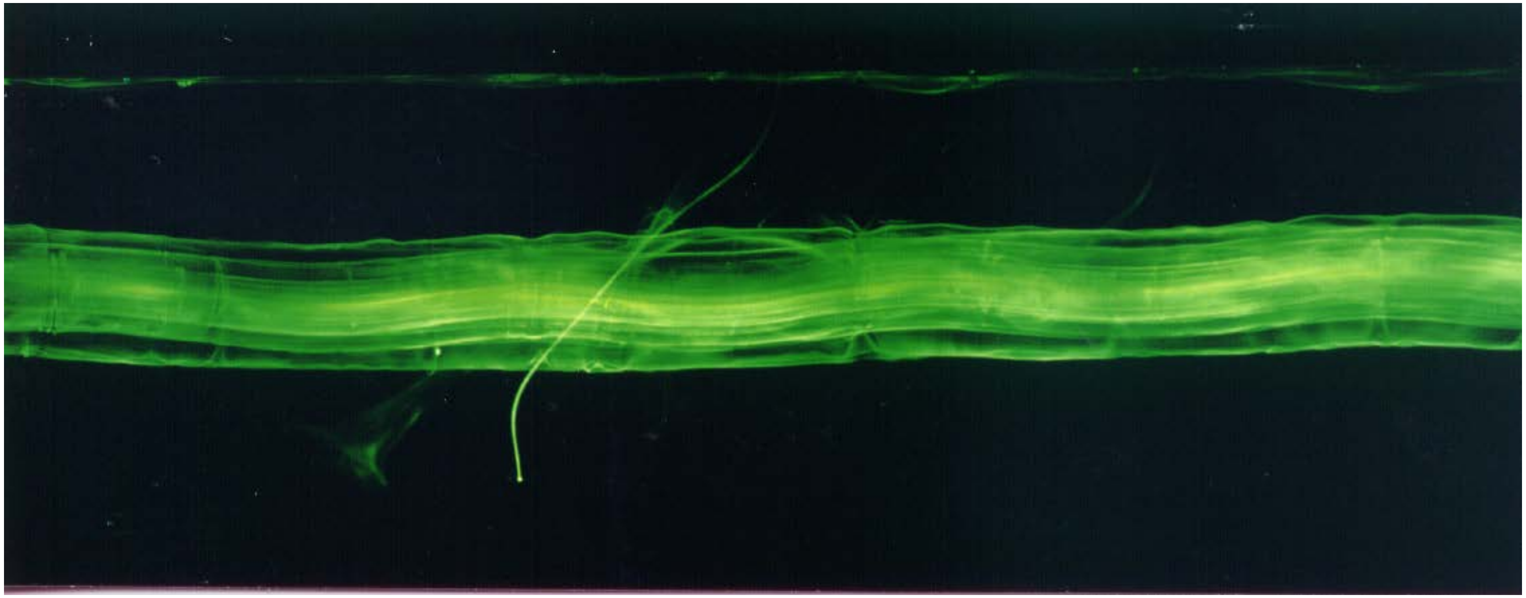




length: 170 cm

Crow instability *(Re = 1500–2500, a/b ≈ 0.2)*

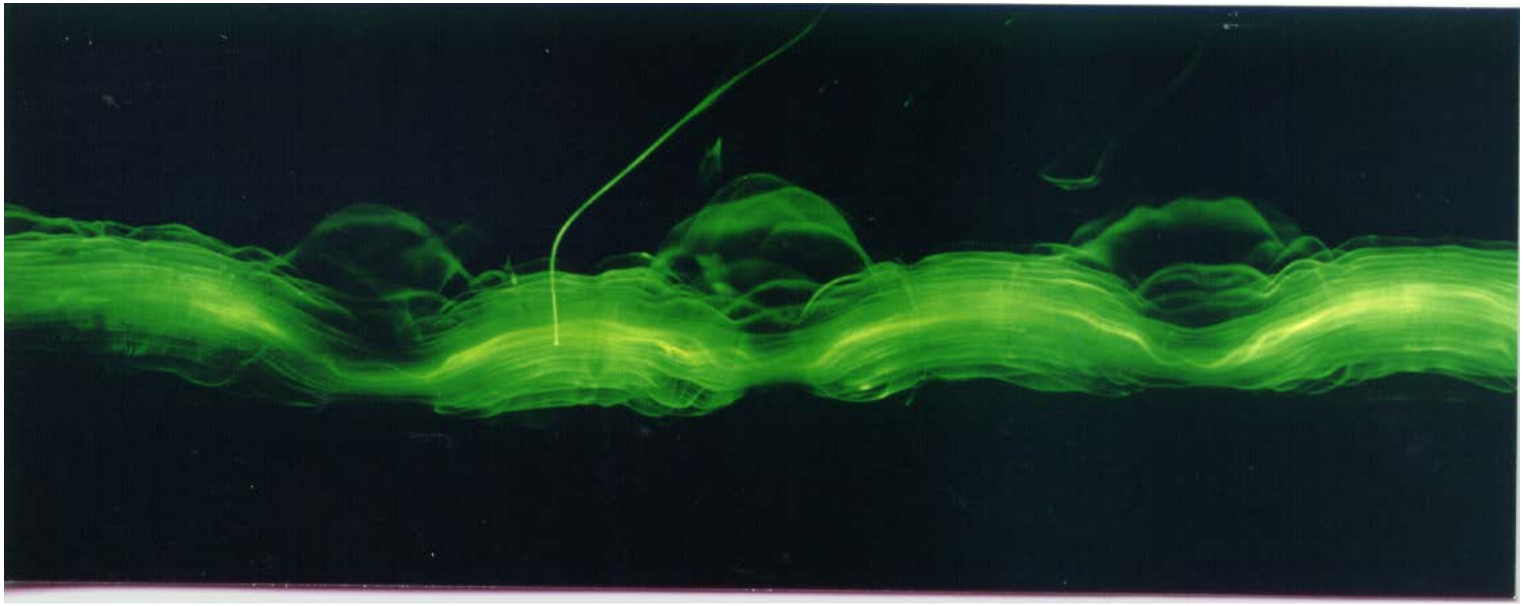
side view



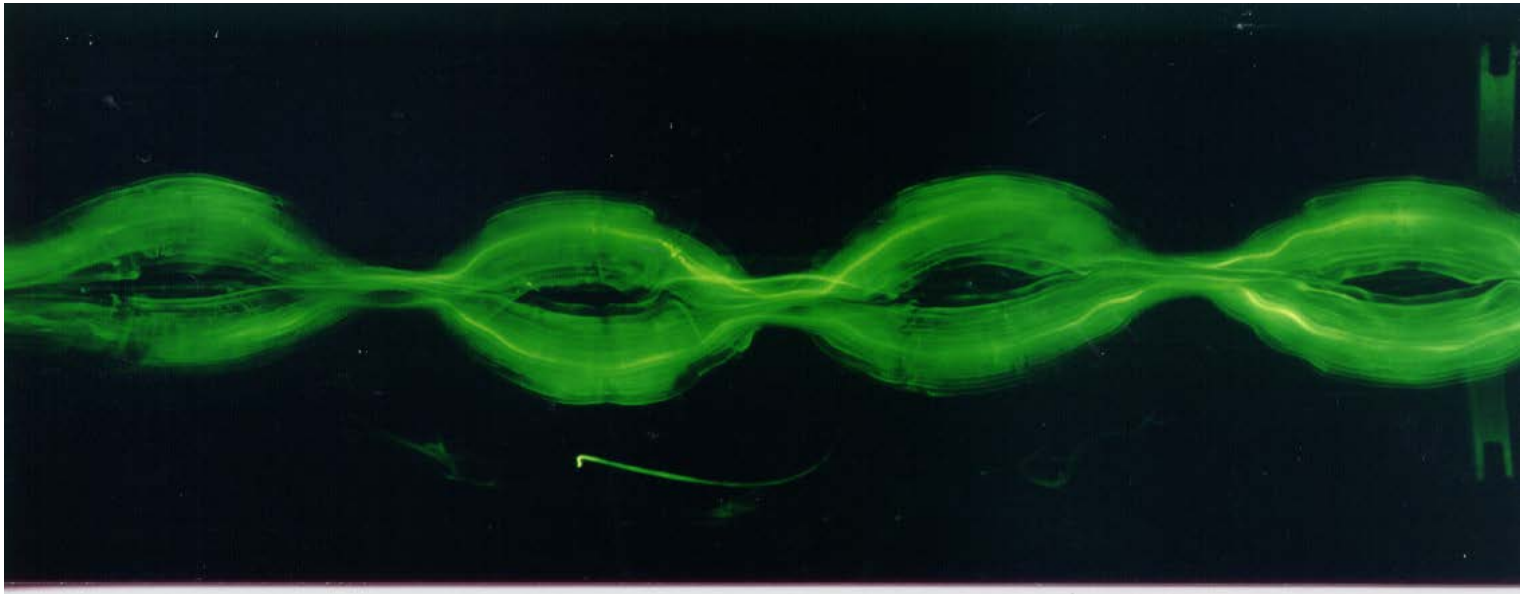
bottom view

Crow instability *(Re = 1500–2500, a/b ≈ 0.2)*

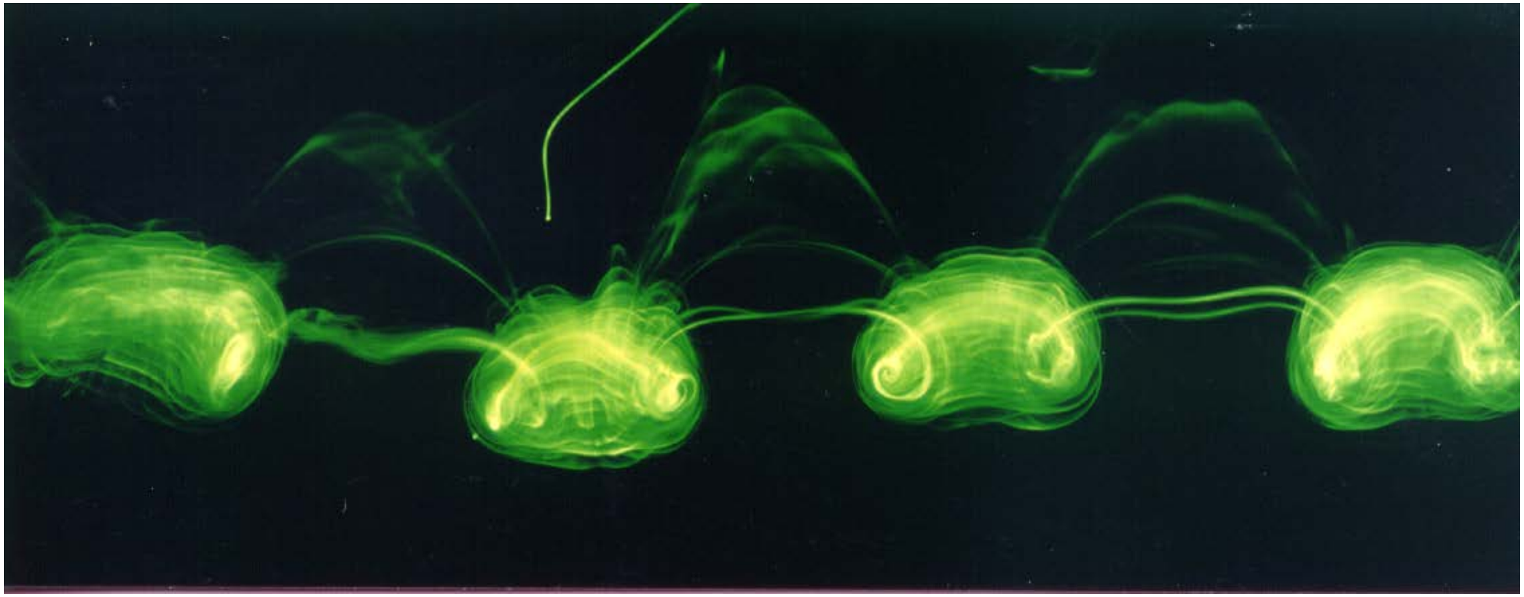
side view



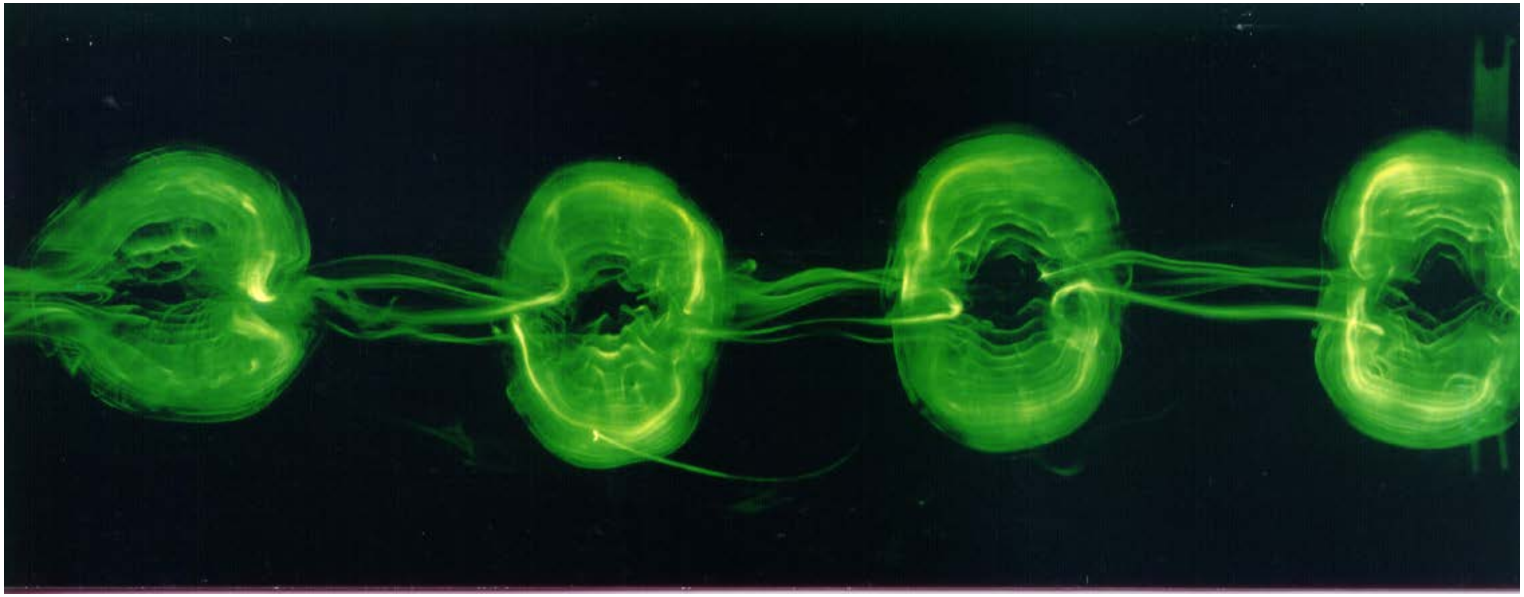
bottom view



Crow instability *(Re = 1500–2500, a/b ≈ 0.2)*



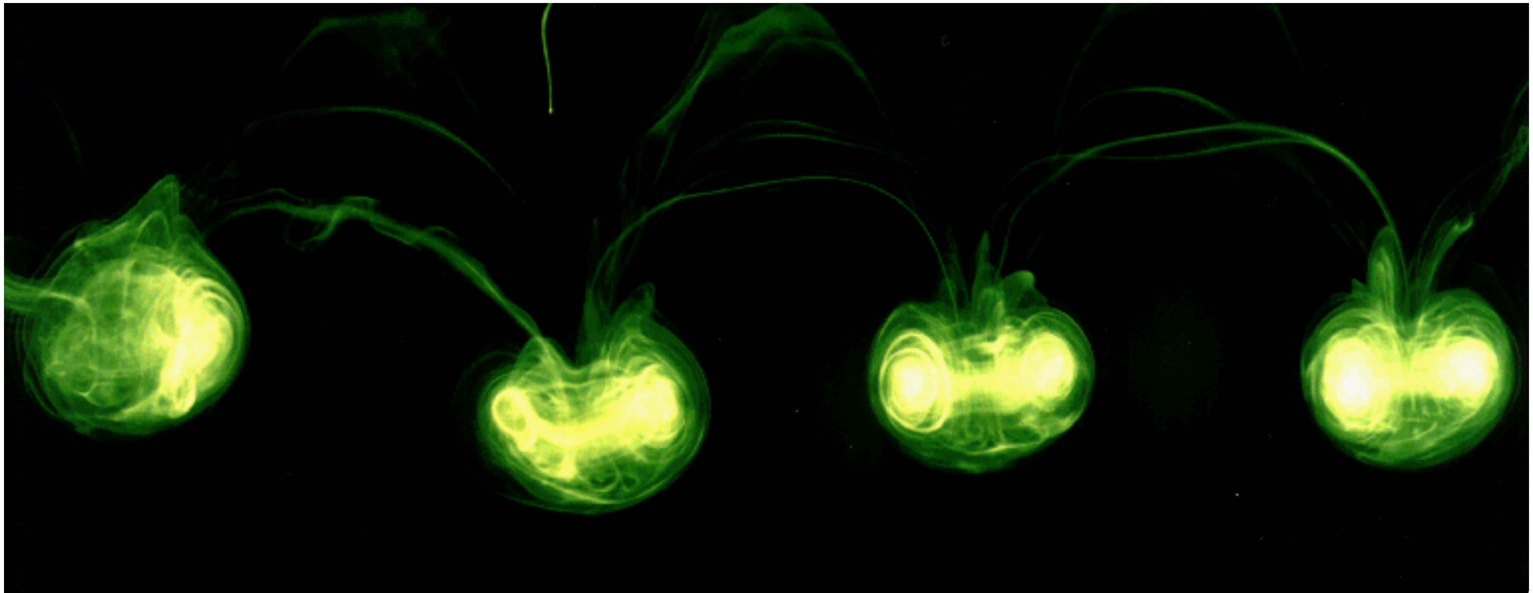
side view



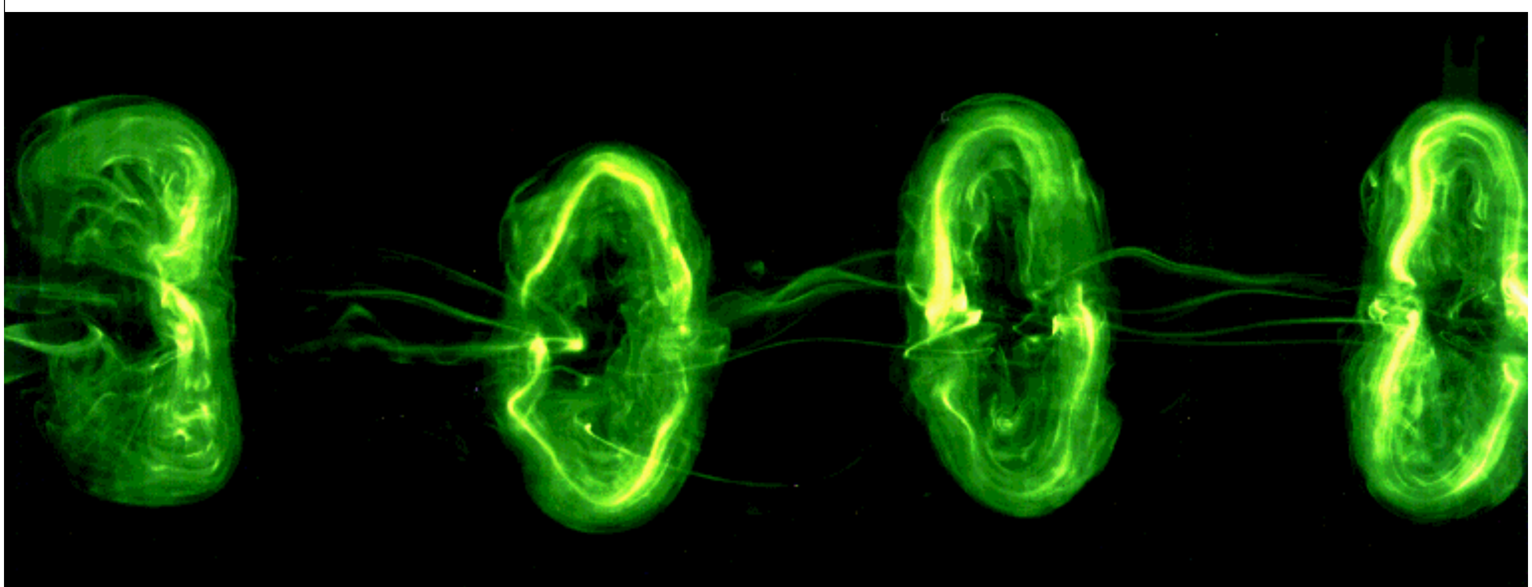
bottom view

Crow instability *(Re = 1500–2500, a/b ≈ 0.2)*

side view

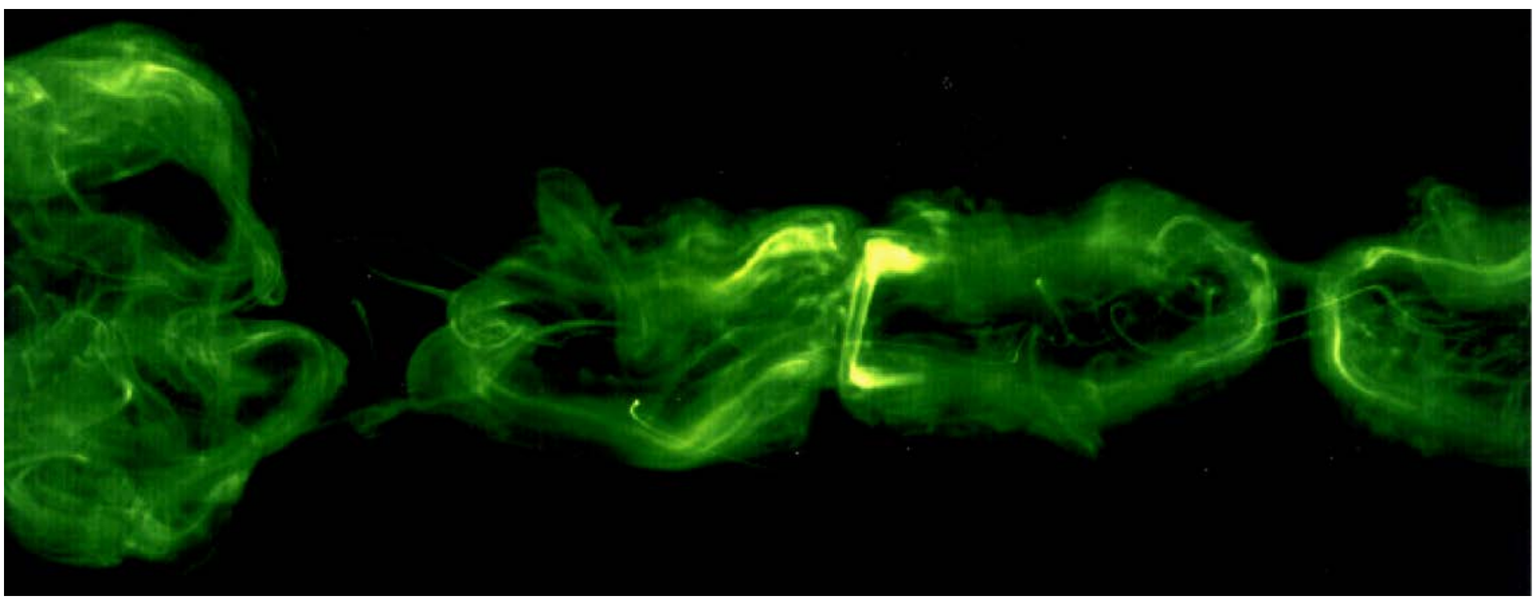


bottom view



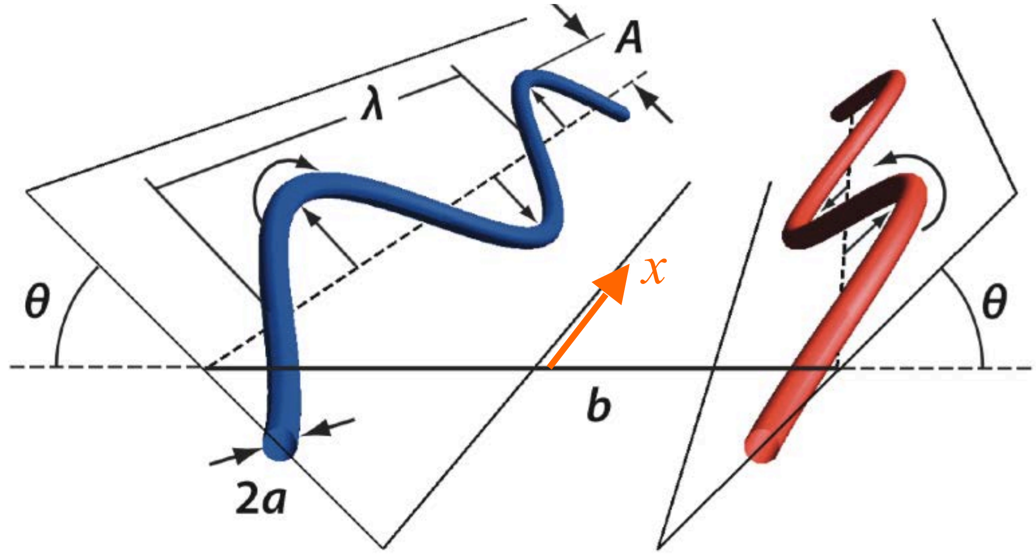
Crow instability *(Re = 1500–2500, a/b ≈ 0.2)*

side view



bottom view

Crow instability mechanism



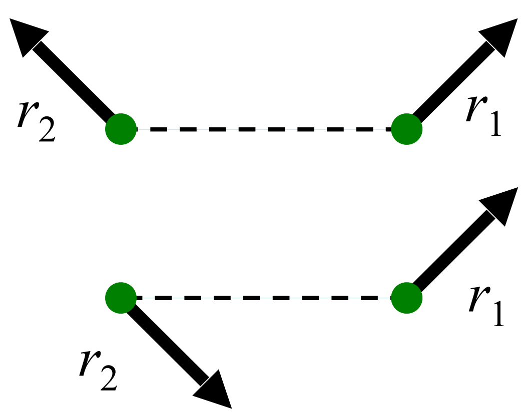
Consider:

- two vortex filaments (core size *a*, separation *b*)

- plane sinusoidal displacement perturbations (wavelength λ, orientation θ)

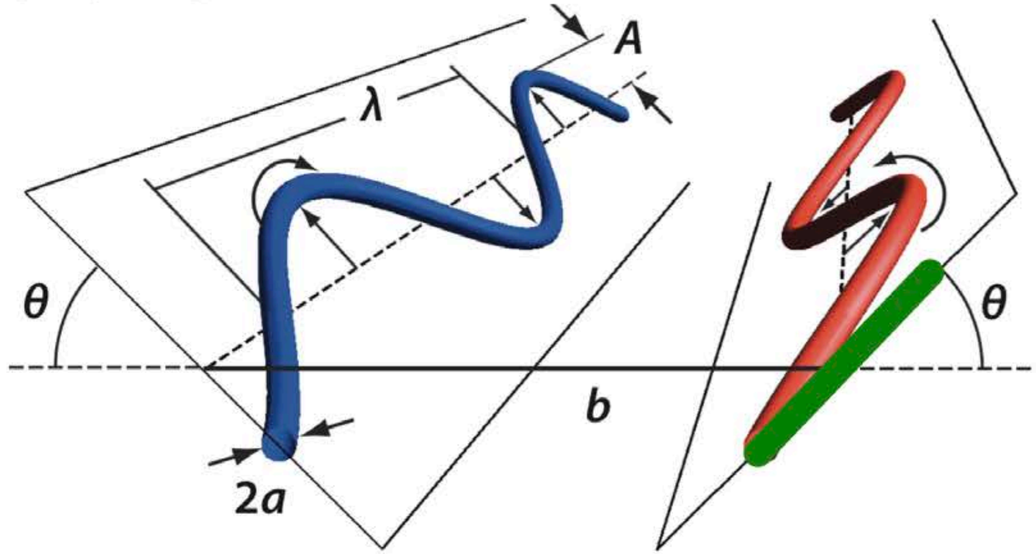
Symmetric modes

Anti-symmetric modes

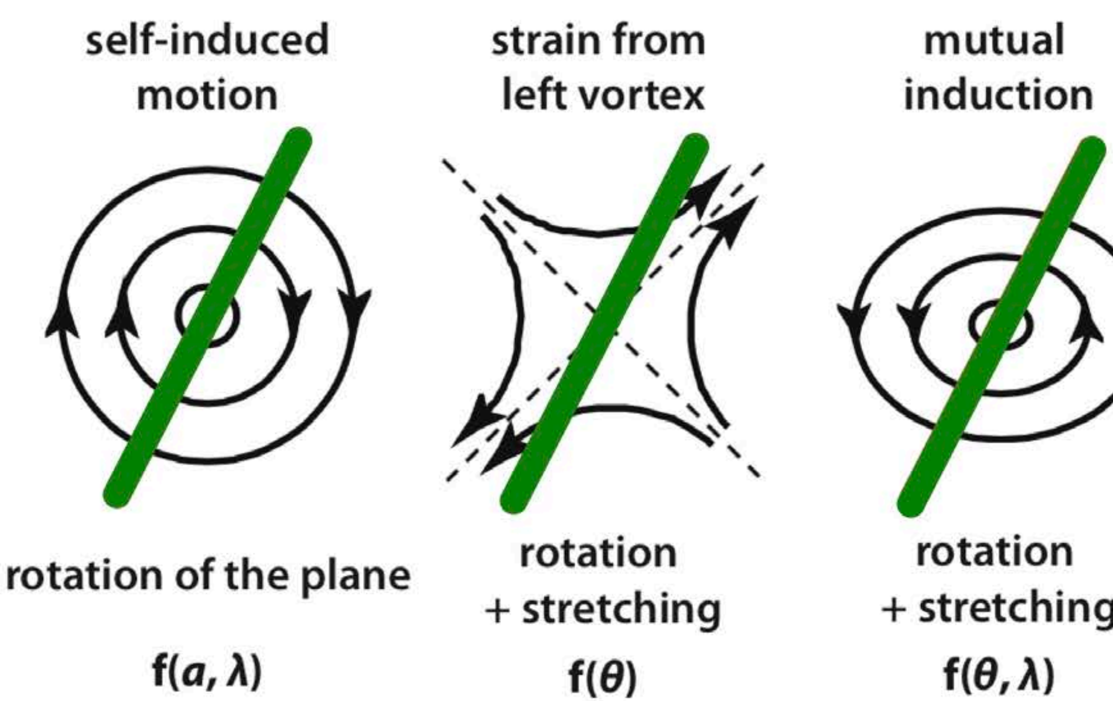


$$\boxed{r_1 = \hat{r}_1 \sin\left(\frac{2\pi x}{\lambda}\right)\exp(\alpha t)}$$

Crow instability mechanism



Evolution of perturbation
(example: **right** vortex):



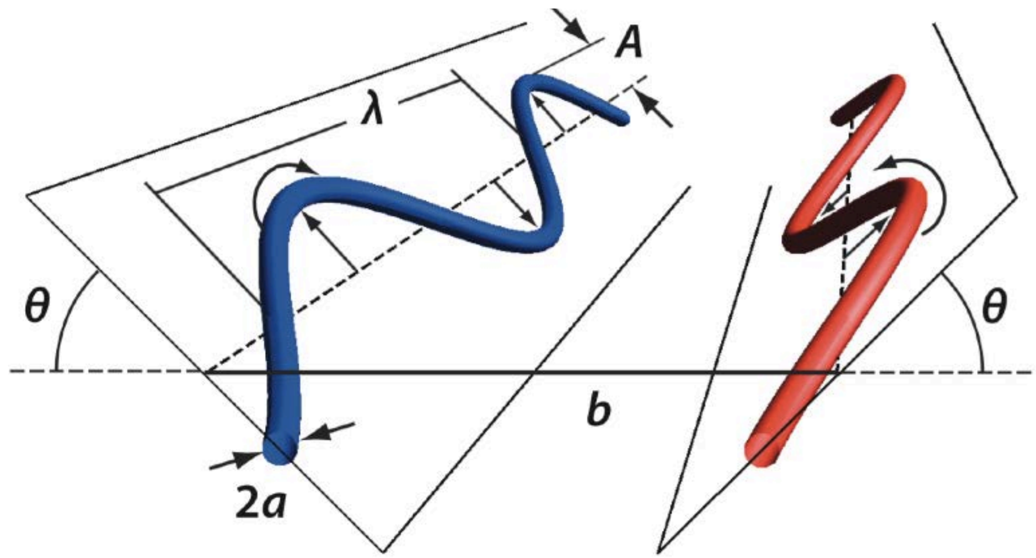
IF

- **total rotation = 0**
- **total stretching > 0**

⇩

Instability

Crow instability mechanism



Find **α :**

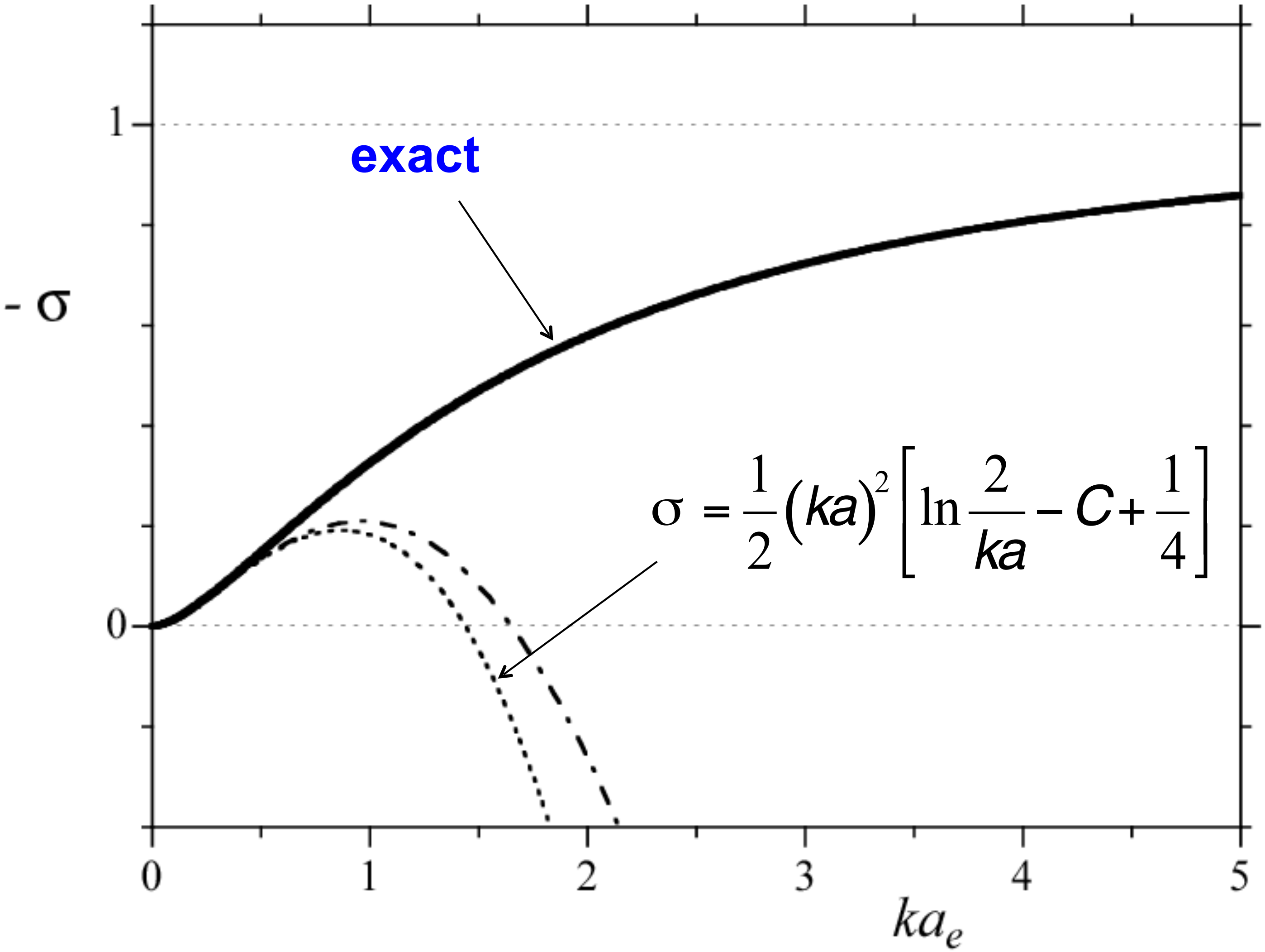
- **Biot-Savart line integrals**
- **cut-off method for self-induction**
- **linearisation**

$$(\alpha^*)^2 = \left[1 - (kb_o)^2 K_o(kb_o) - kb\,K_1(kb_o) - \frac{\sigma}{(a_e/b_o)^2}\right]\left[1 + kb_o\,K_1(kb_o) + \frac{\sigma}{(a_e/b_o)^2}\right]$$

$\alpha^* = \alpha\,(2\pi b^2/\Gamma)$ $\quad$ $k = 2\pi/\lambda$ $\quad$ σ: self-induced rotation rate

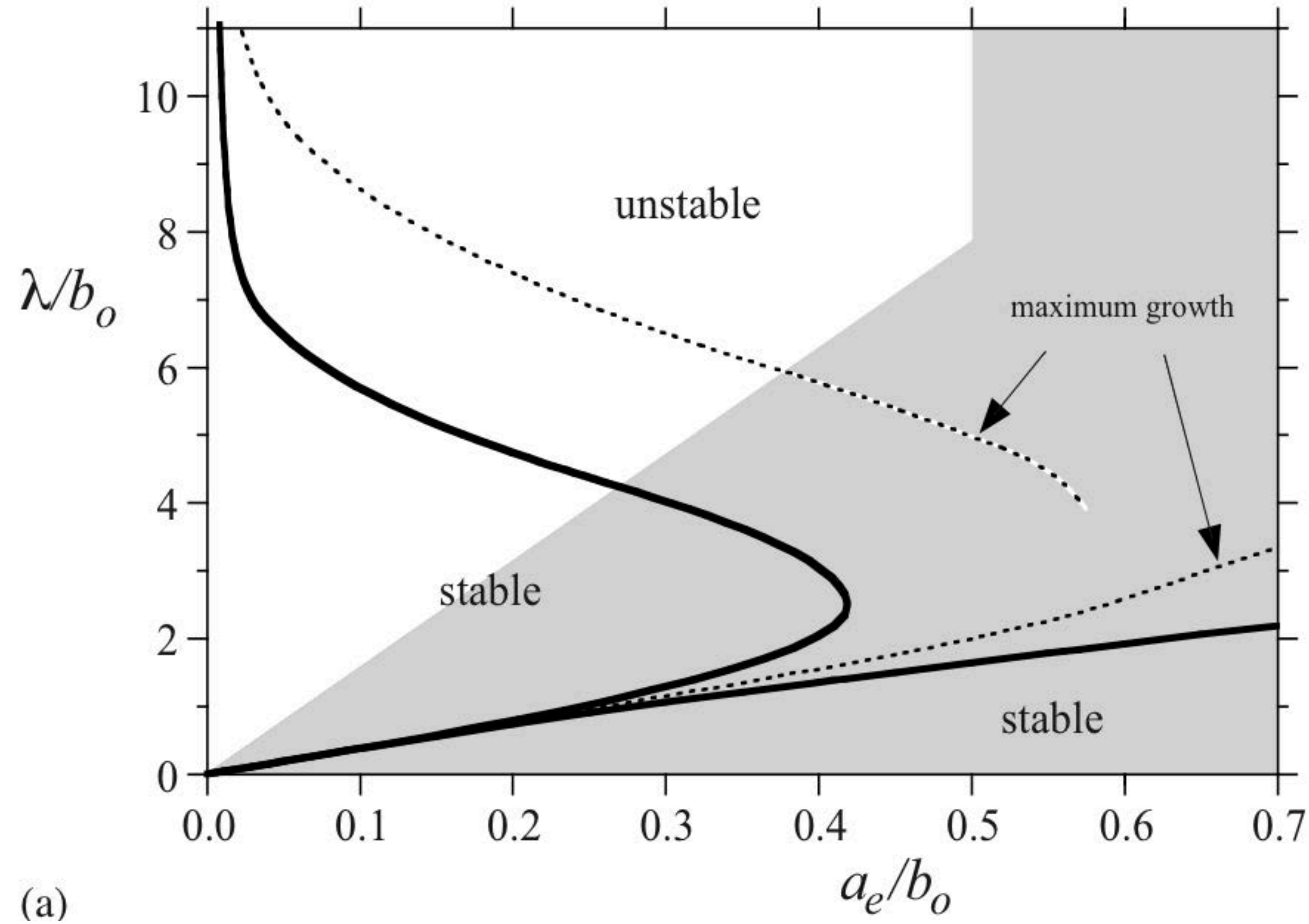
Self-induced rotation of a sinusoidal vortex filament

(Rankine vortex, Kelvin 1880)



Stability diagram for a pair of counter-rotating vortices

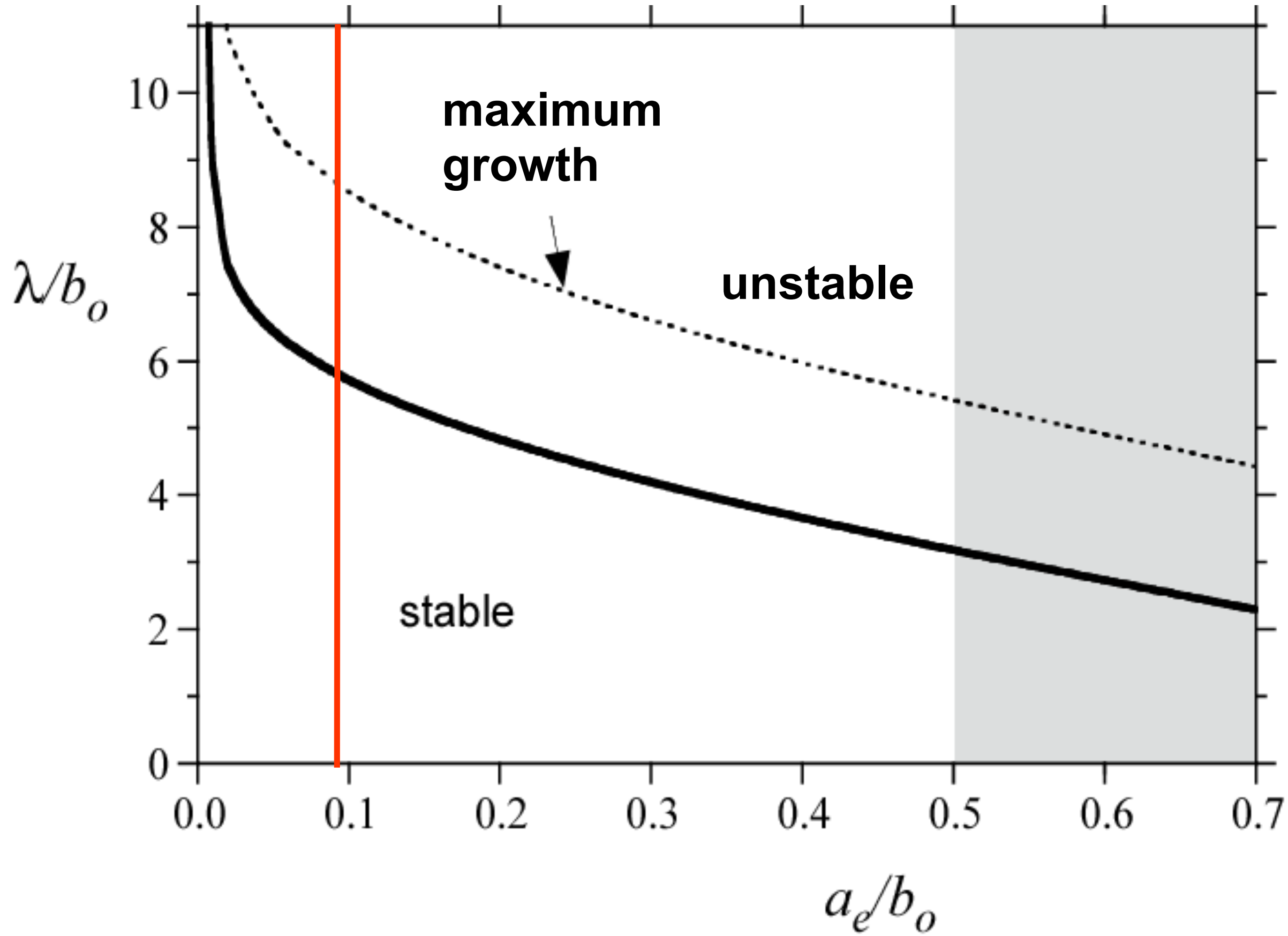
(symmetric mode)



(a)

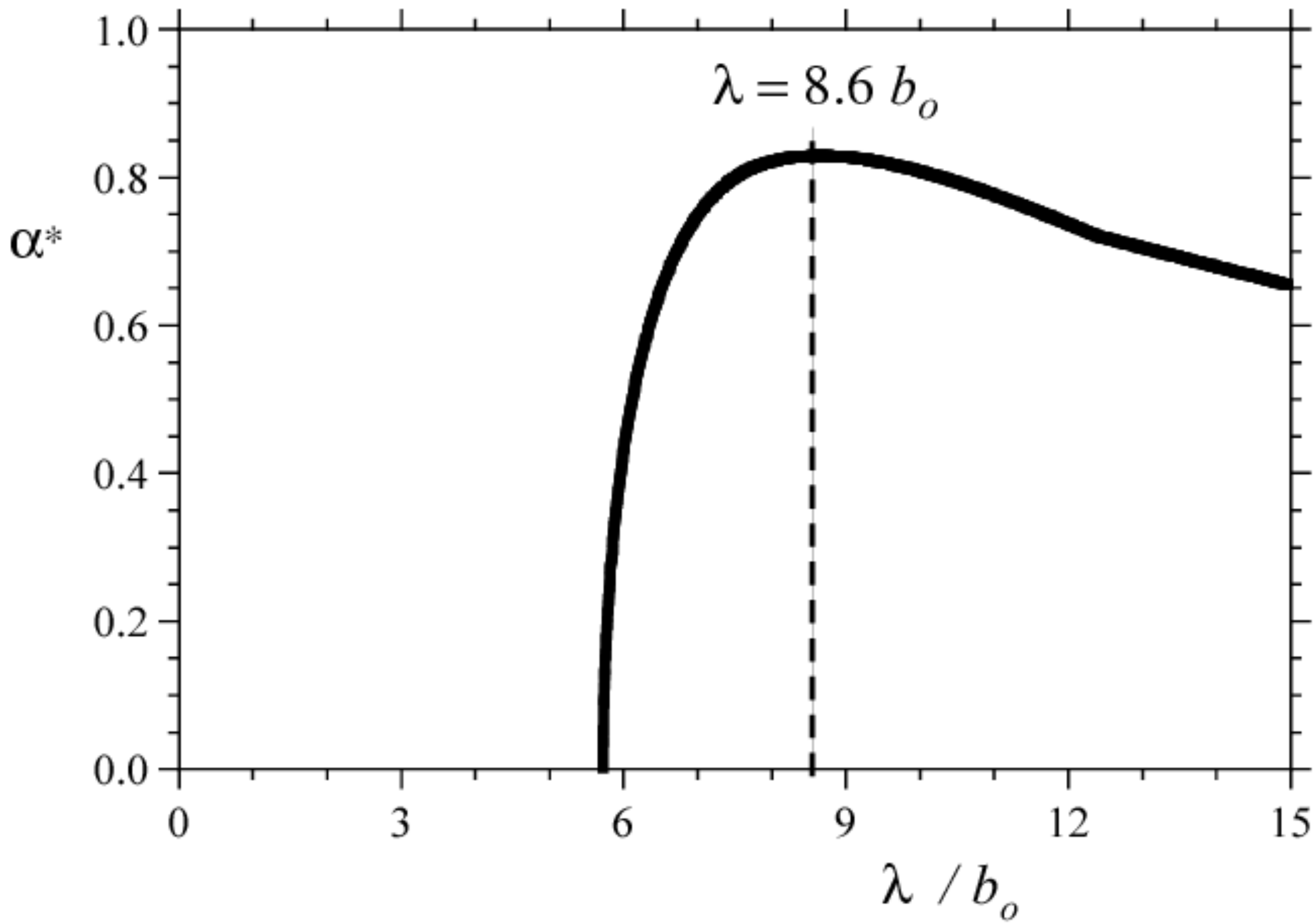
Stability diagram for a pair of counter-rotating vortices

(symmetric mode)



Growth rate of the Crow instability

(a/b = 0.0985)



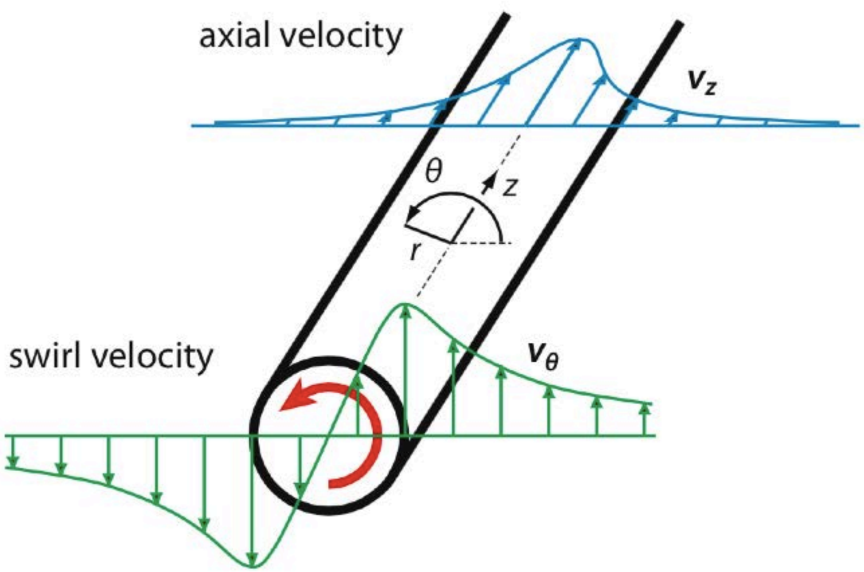
Generalization to vortices other than Rankine

Equivalent core size
(Widnall *et al.* 1971)

$$a_e = a \cdot \exp\left[\frac{1}{4} - A + C\right]$$

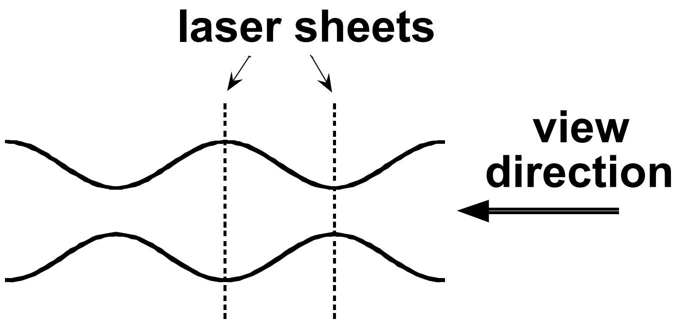
$$A = \lim_{r\to\infty}\left[\frac{4\pi^2}{\Gamma^2}\int_0^r \overline{r}v_\phi^2(\overline{r})d\overline{r} - \ln\frac{r}{a}\right]$$

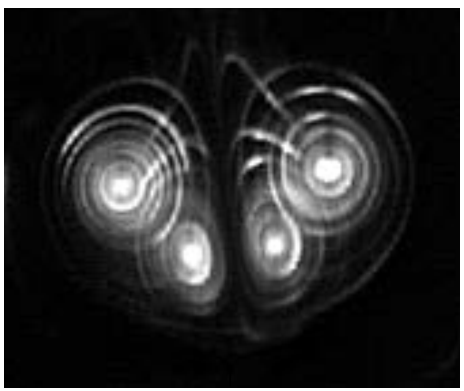
$$C = \frac{8\pi^2}{\Gamma^2}\int_0^\infty \overline{r}v_z^2(\overline{r})d\overline{r}$$

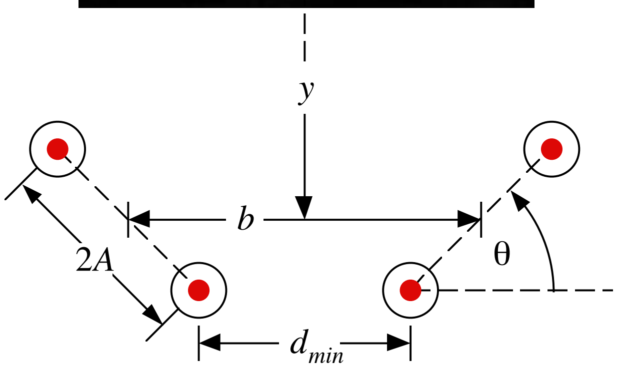


valid for $\lambda \gtrsim 15\ a$

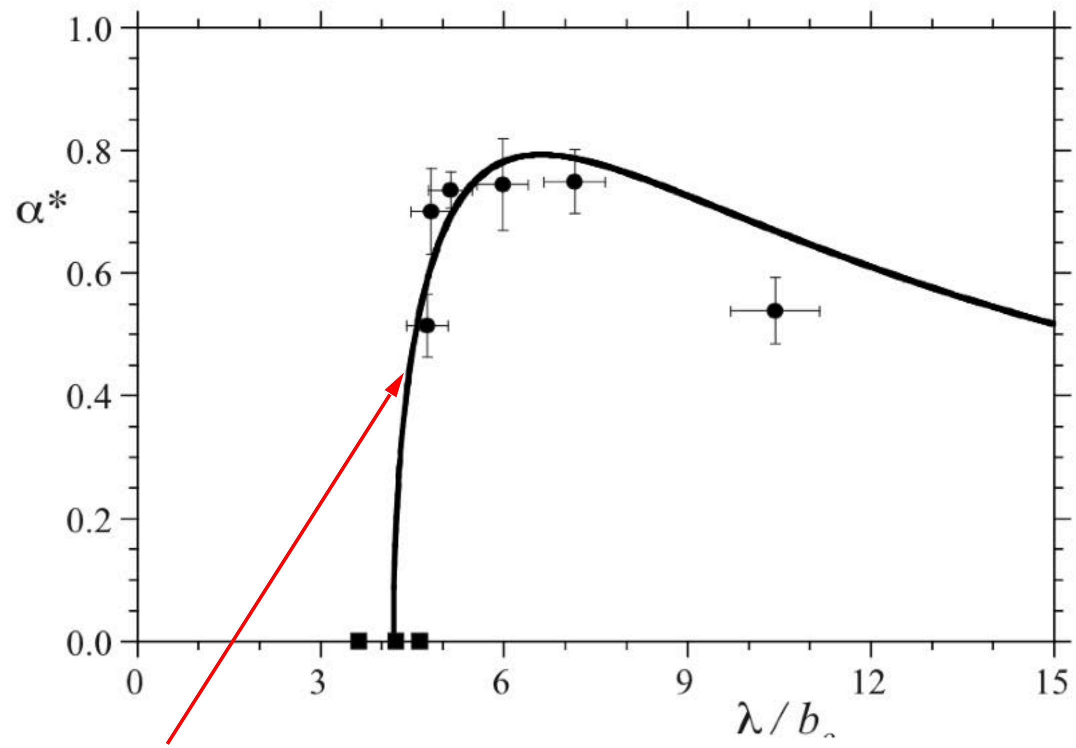
Crow instability *(measurements)*







instability growth rate

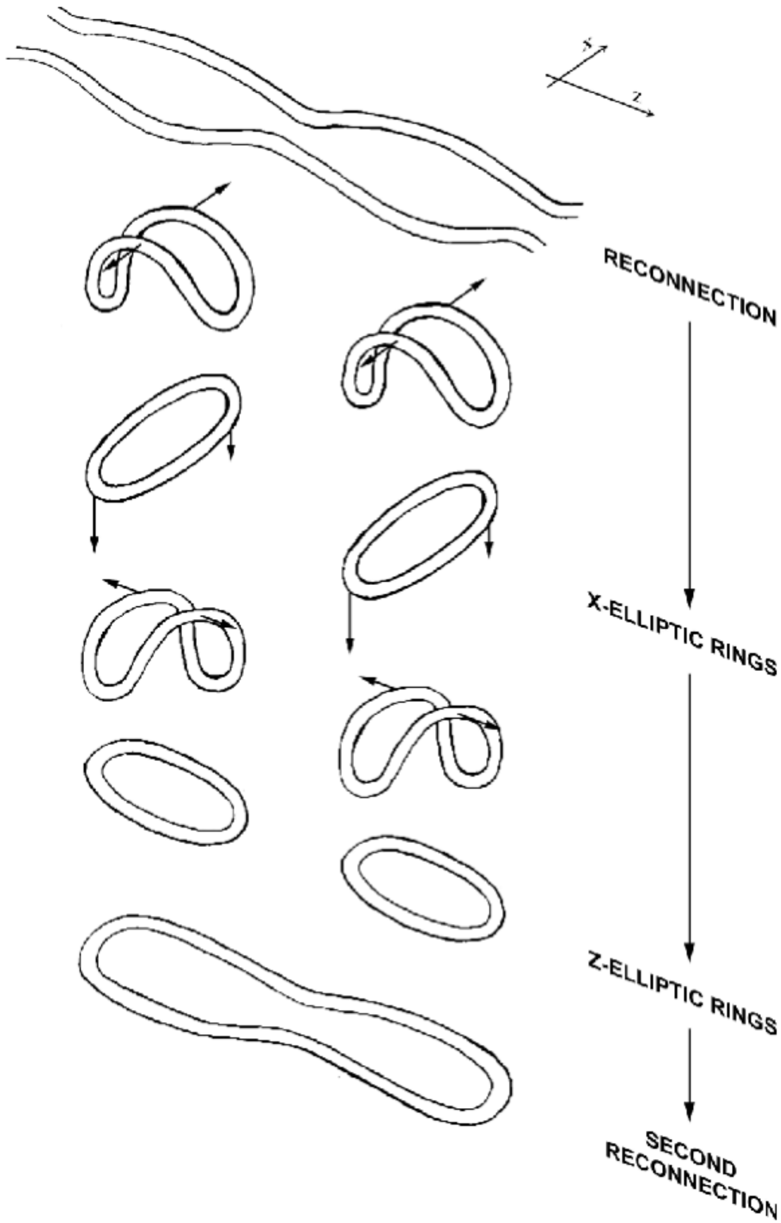


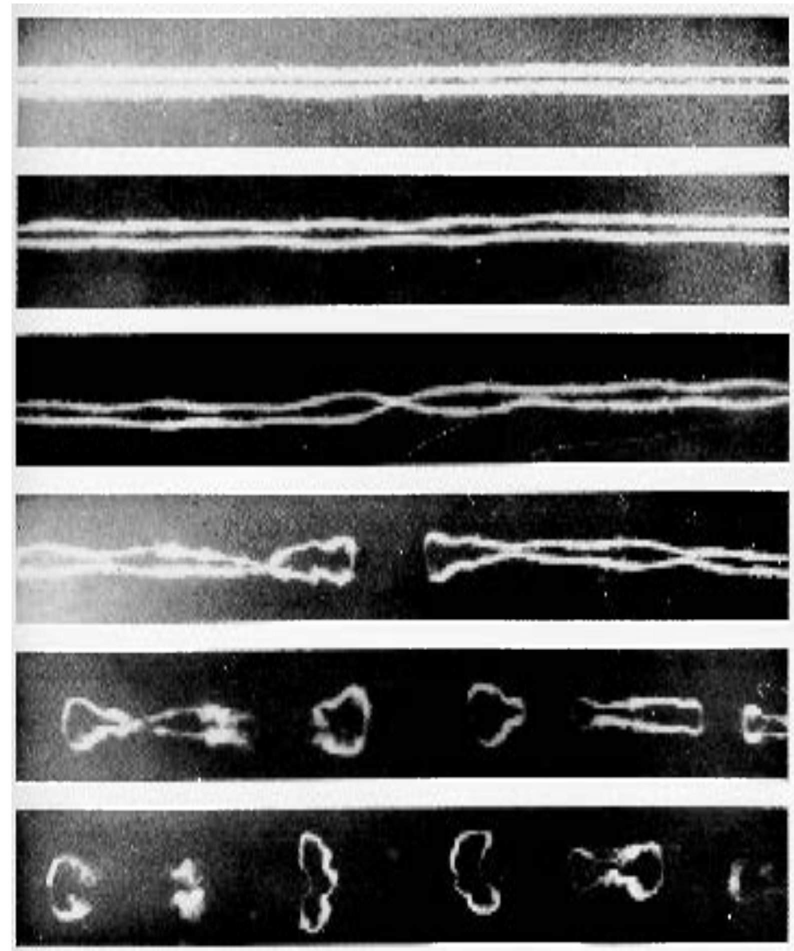
theoretical prediction

- uses ***a*/*b*$_o$** and vortex **velocity profile**

(Crow 1970, Widnall *et al.* 1971)

Crow instability *(long-term evolution)*





Photographs of aircraft wake (Crow 1970)

Tip vortices of a Hawker Sea Fury

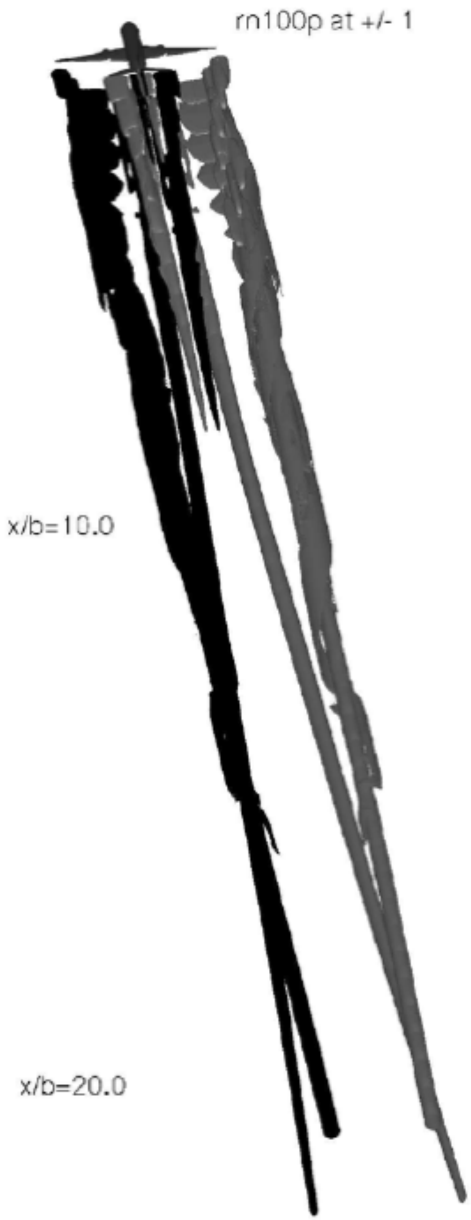
AIR14 air show at Payern Switzerland, 7 September 2014

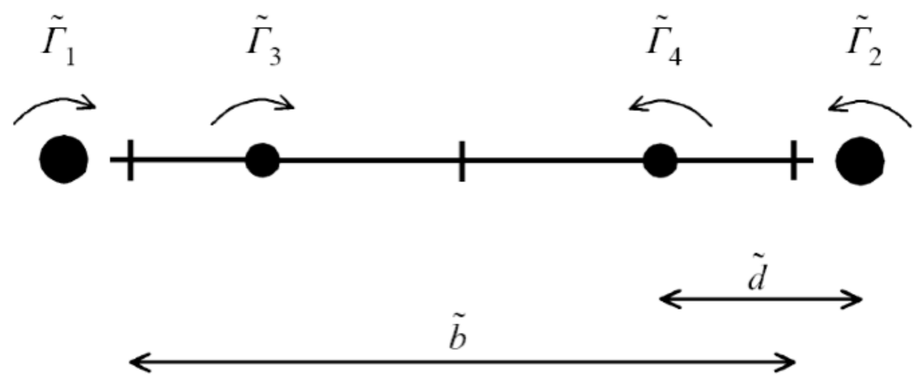
(video by G. Balestra)

Long-wave instability
Vortex pairs

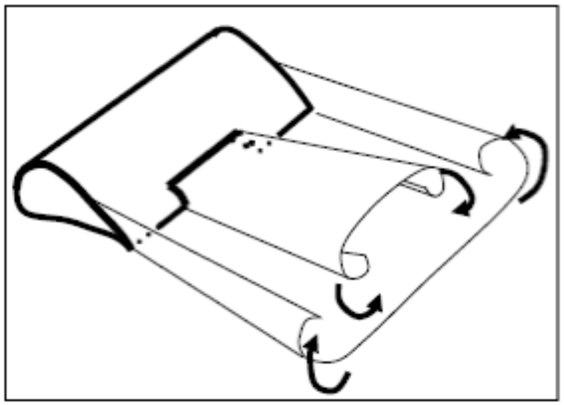
“Crow” instability of 4-vortex systems

Co-rotating 4-vortex system
(Crouch, Boeing)

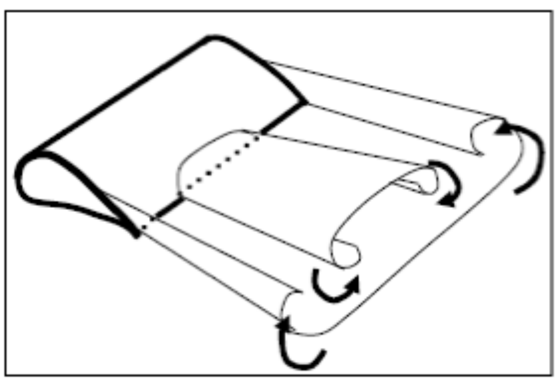




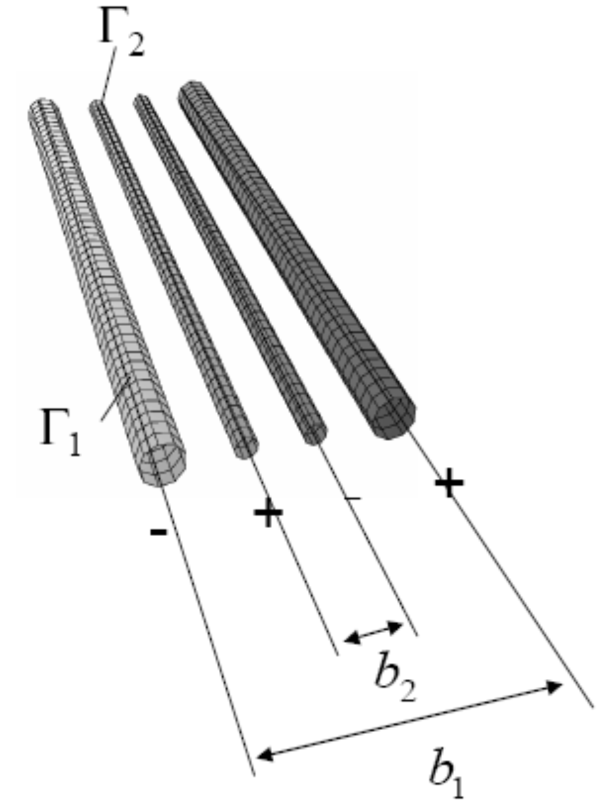
Counter-rotating 4-vortex system
(Jacquin et al., ONERA)



- outboard flaps
- horizontal tail plane

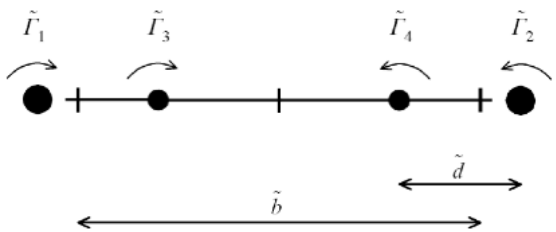


- C_{zmax} : separation !

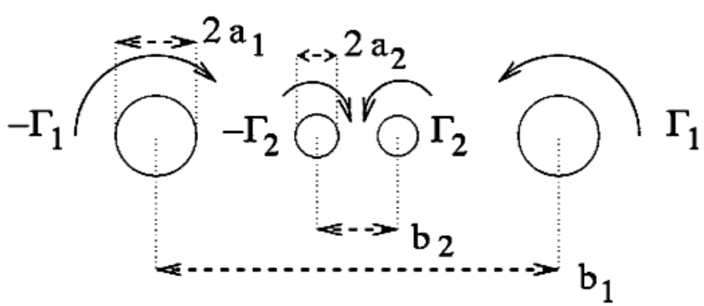


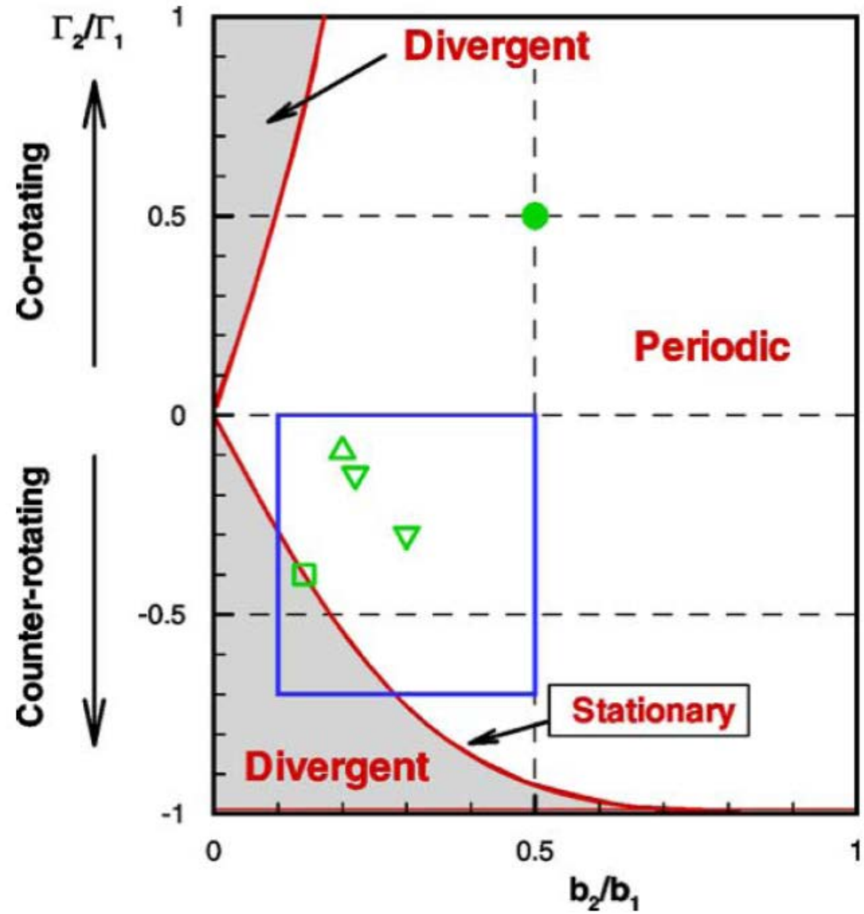
Four-vortex systems

• **two co-cotating pairs**



• **two counter-cotating pairs**

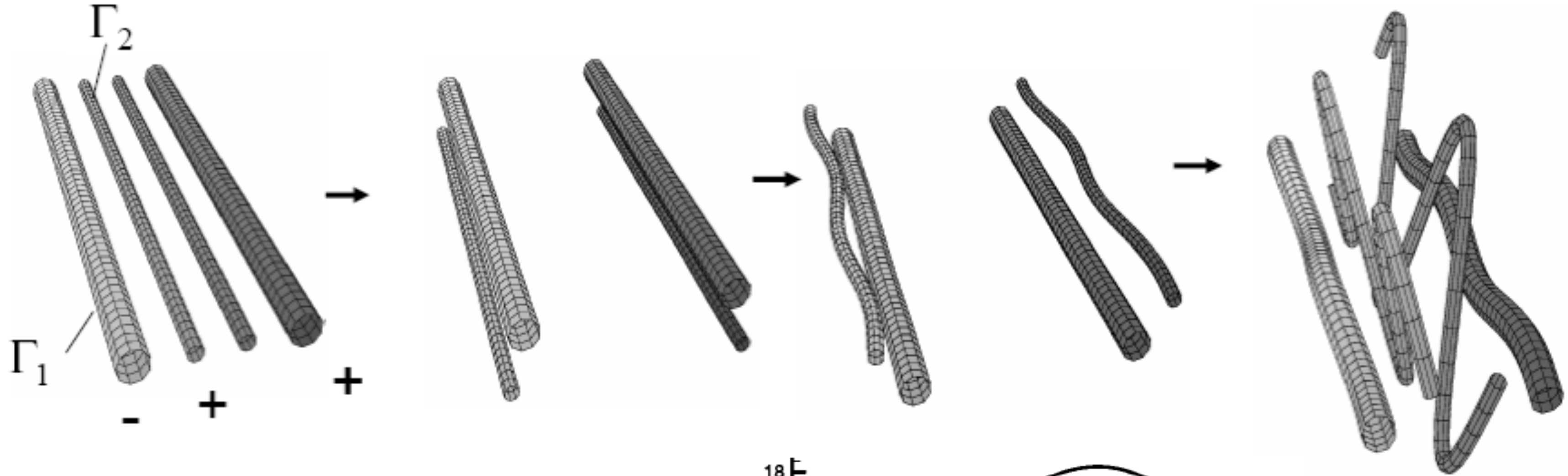




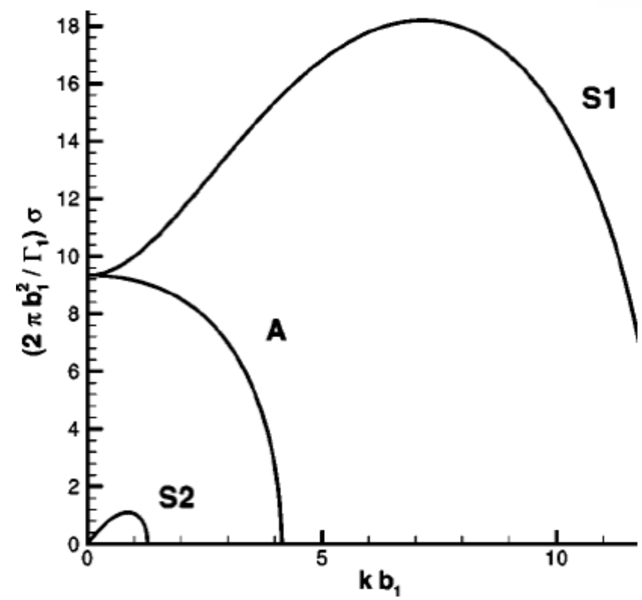
classification chart

(Fabre *et al.* 2002)

Counter-rotating 4-vortex system
(Jacquin et al., ONERA)



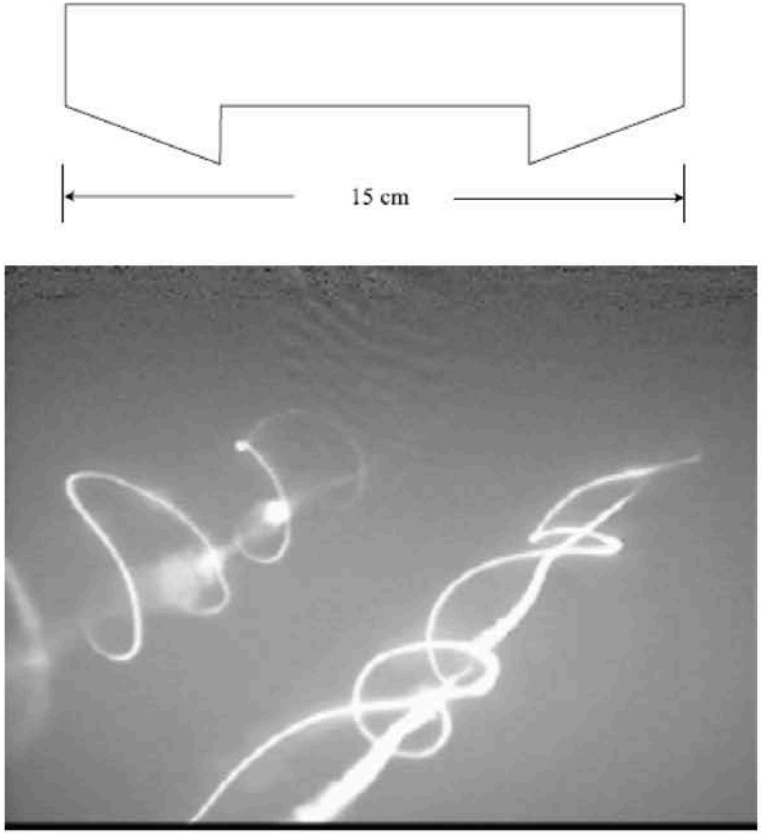
- **deformations of inner vortices**
- **large growth rate**
- λ/b_1 **= O(1)**



Counter-rotating 4-vortex system

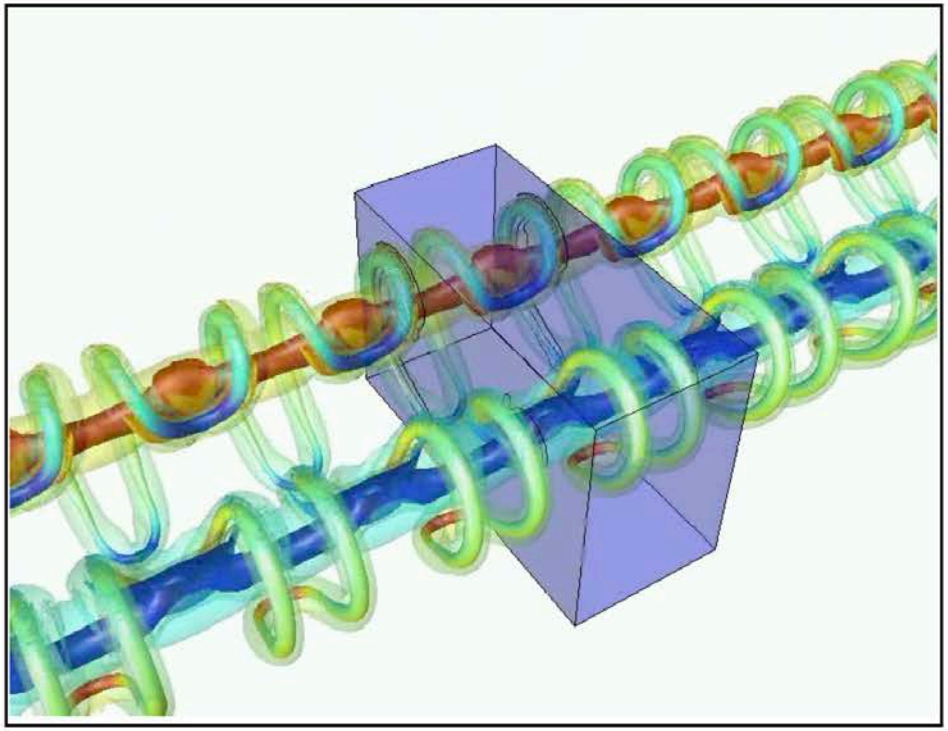
Towing tank experiment

Ortega, Savas (Berkeley)



Numerical simulation

Winckelmans (UCL)

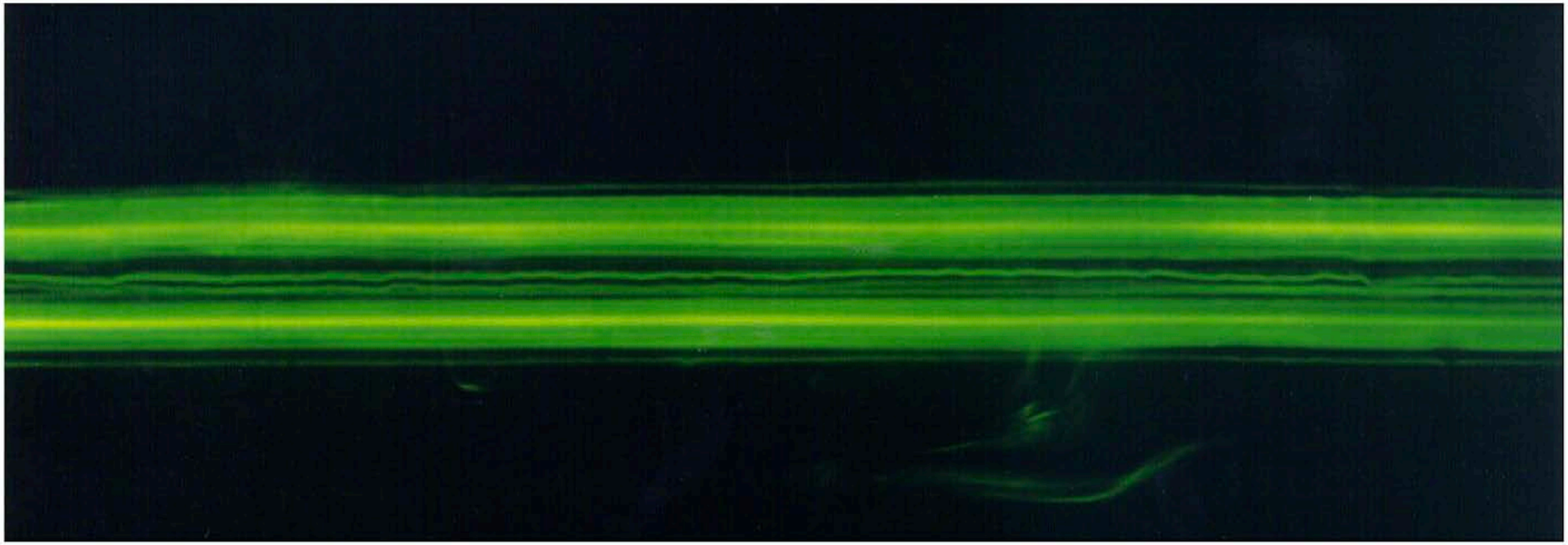


Short-wave instability
Vortex pairs

Elliptic instability

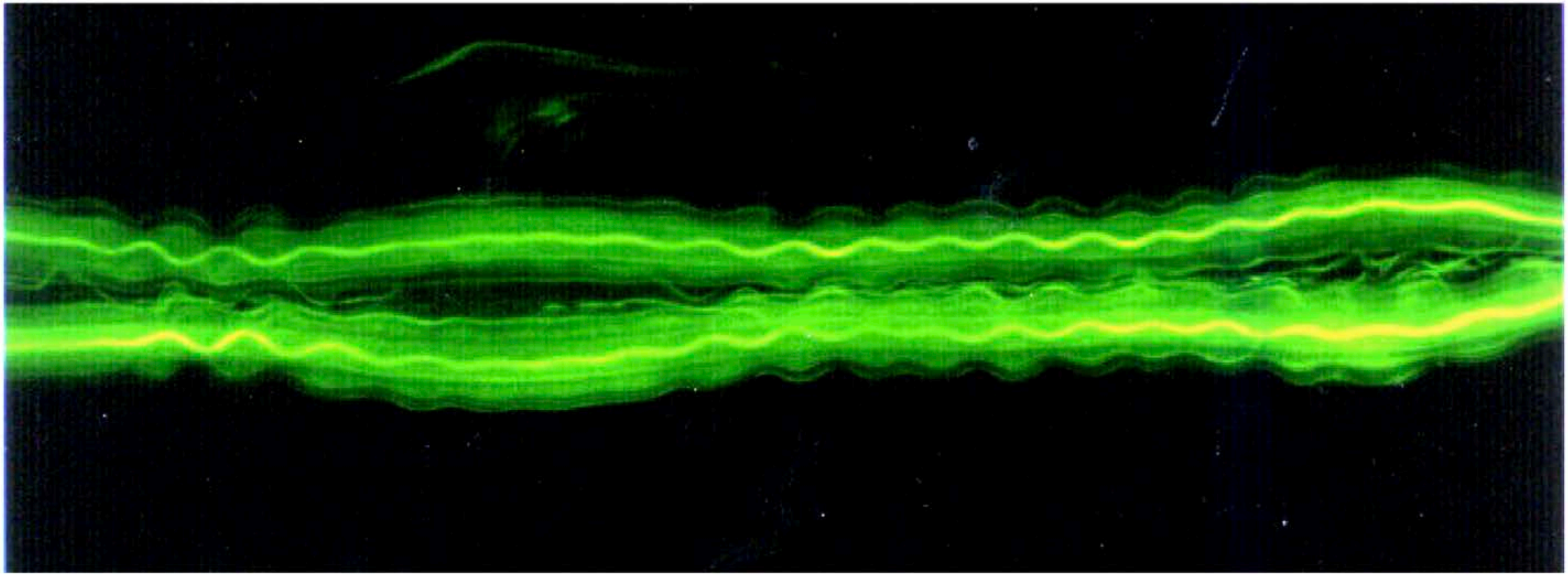
Short-wave instability *(Re = 2500–4000, a/b ≈ 0.2)*

initial condition



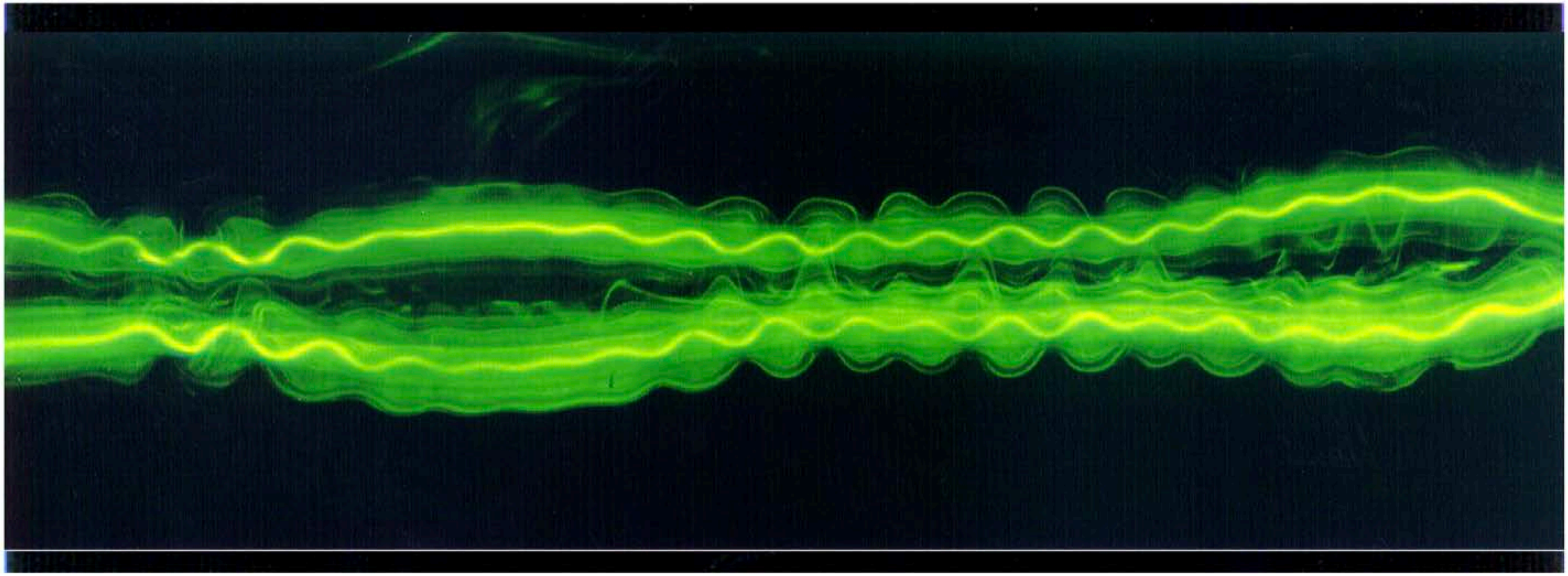
bottom view

Short-wave instability *(Re = 2500–4000, $a/b \approx 0.2$)*



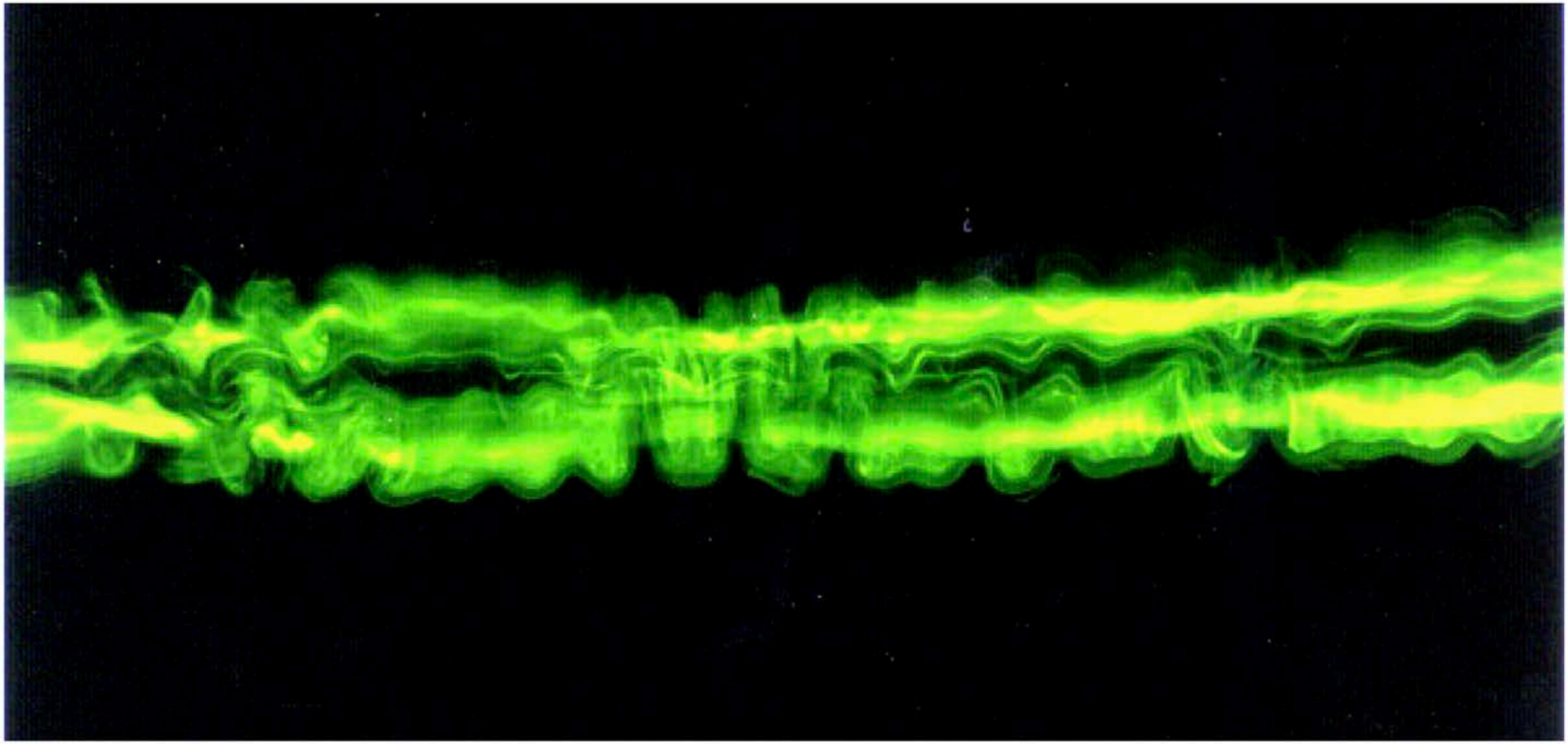
bottom view

Short-wave instability *(Re = 2500–4000, a/b ≈ 0.2)*



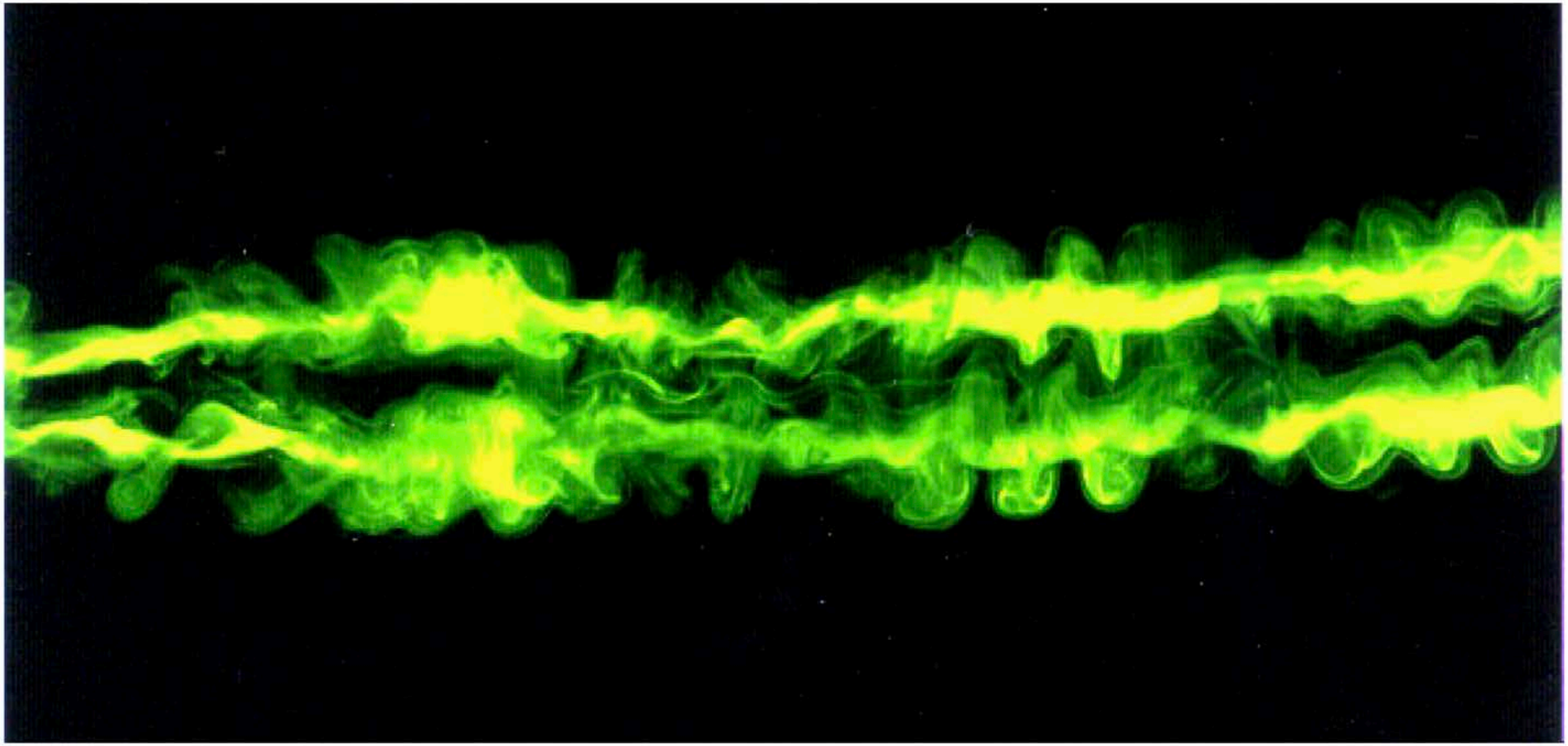
bottom view

Short-wave instability *(Re = 2500–4000, a/b ≈ 0.2)*



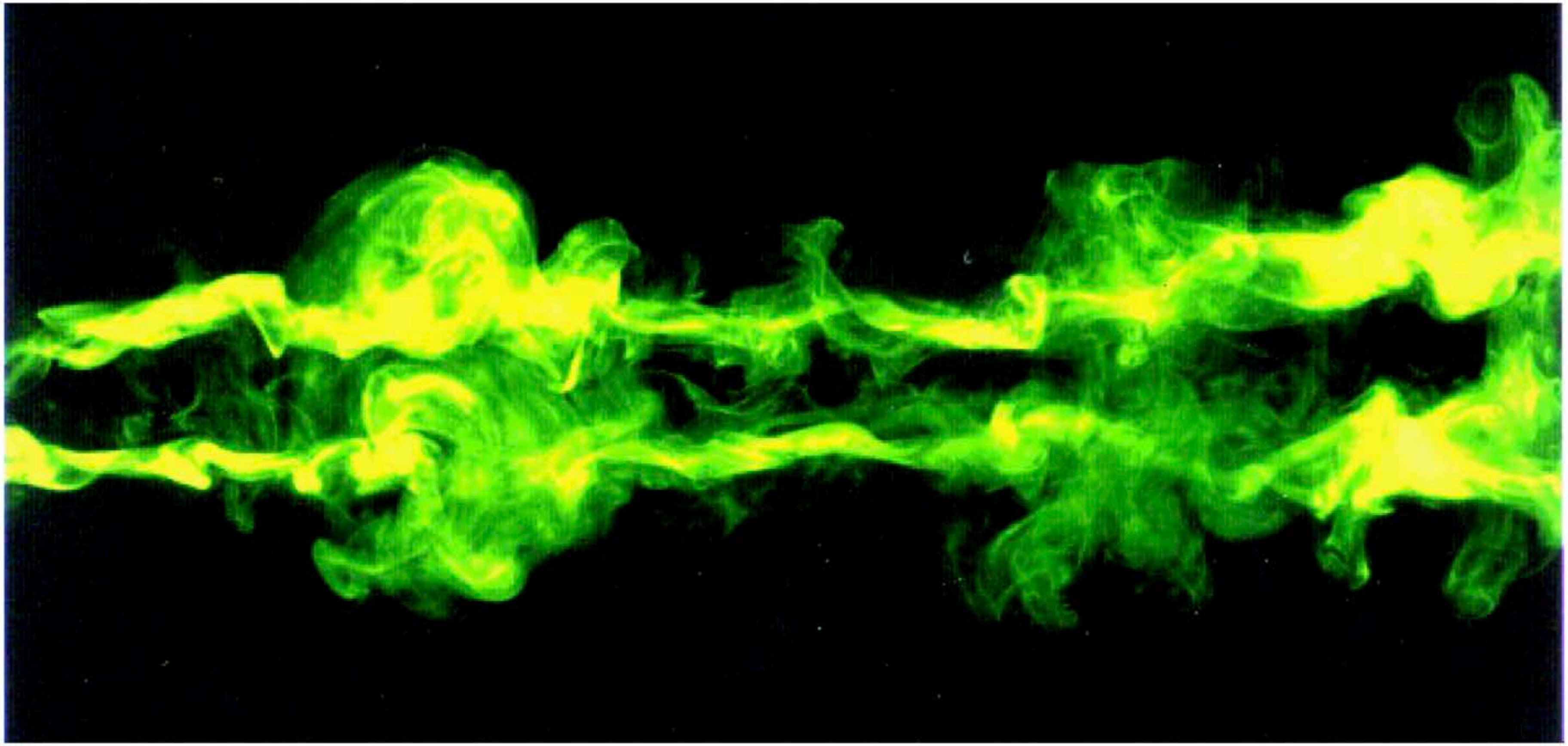
bottom view

Short-wave instability *(Re = 2500–4000, a/b ≈ 0.2)*



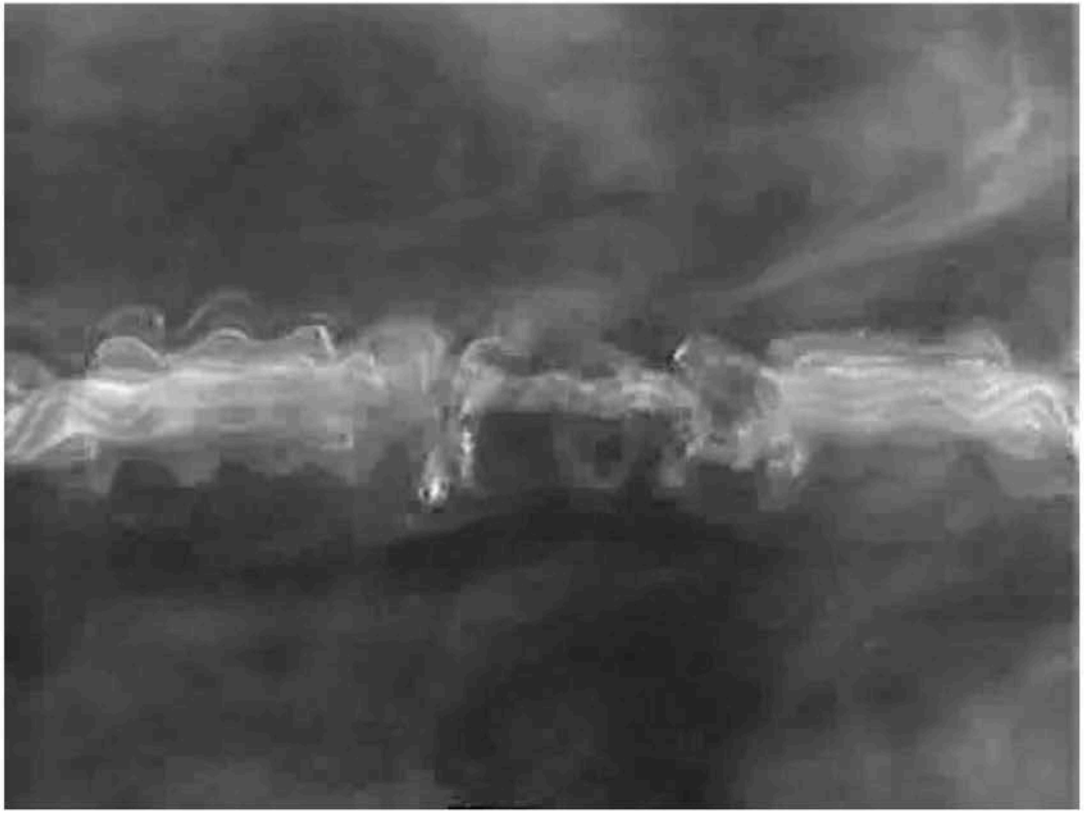
bottom view

Short-wave instability *(Re = 2500–4000, a/b ≈ 0.2)*



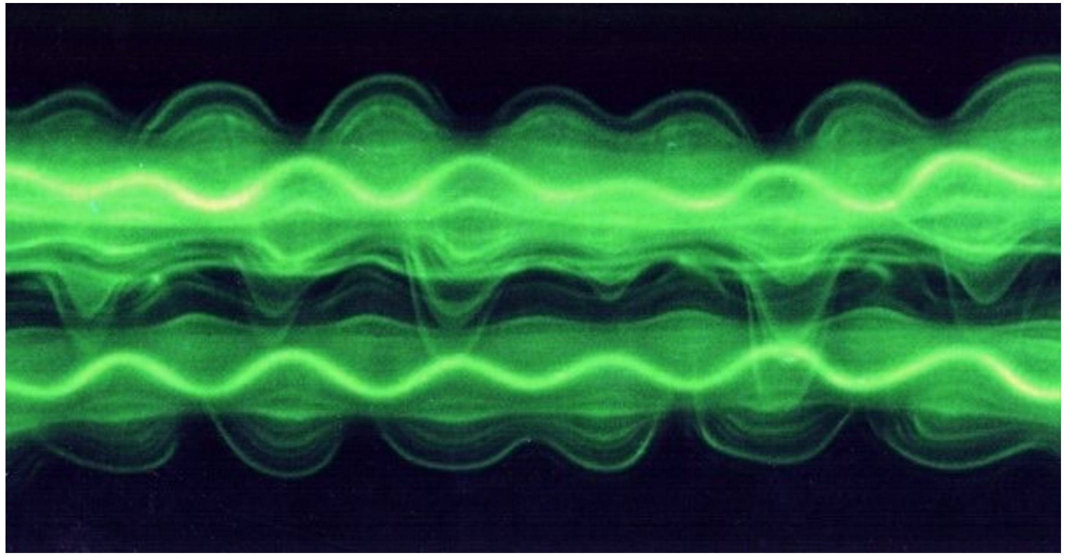
bottom view

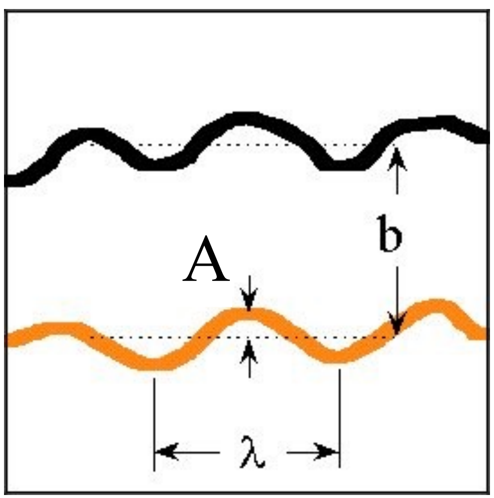
Short-wave instability *(Re = 2500–4000, a/b ≈ 0.2)*



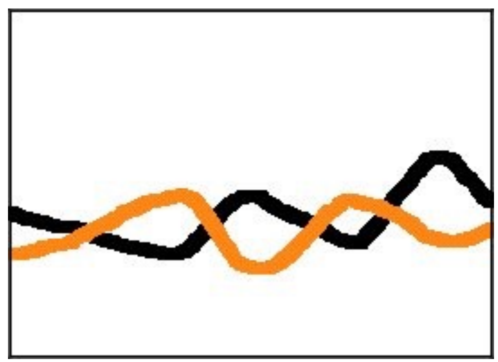
side view

Short-wave instability *(spatial structure & symmetry)*

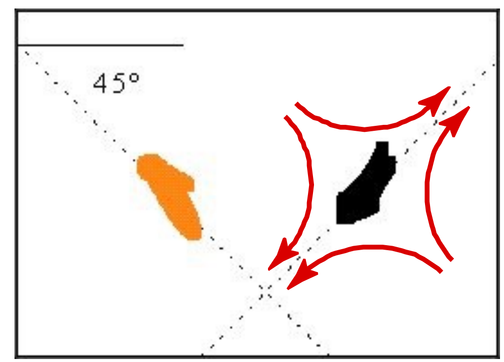




bottom view



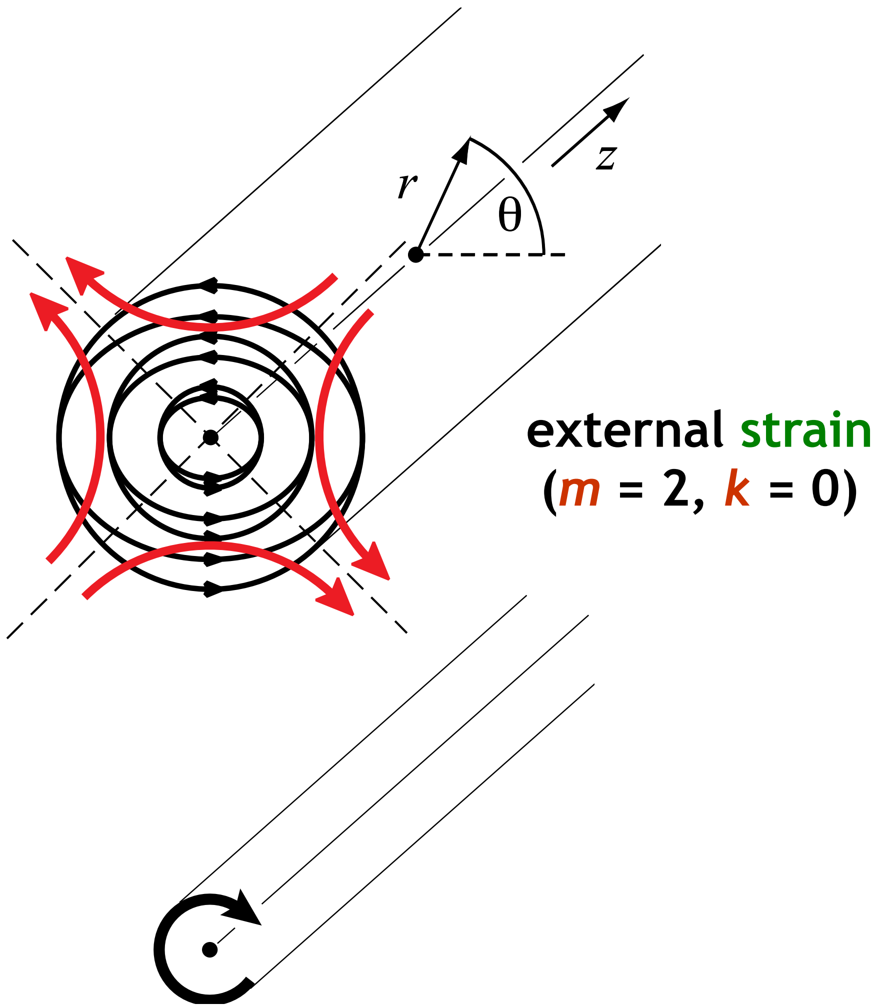
side view



axial view

Short-wave instability mechanism

example: vortex in a strain - elliptic instability



Perturbations of a vortex
("Kelvin modes")

$u(r) \cdot \exp[i(kz + m\theta + \omega t)]$

↓

Resonance between two modes (m_1, k_1, ω_1) and (m_2, k_2, ω_2) and a **strain field** is possible if

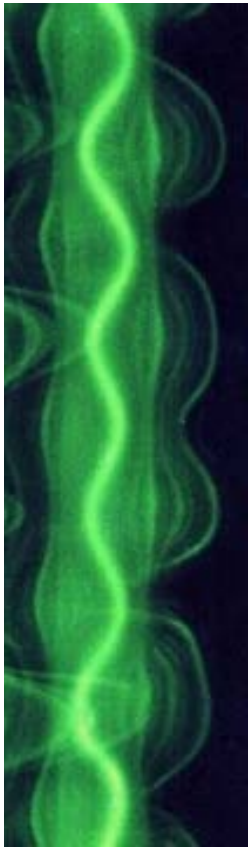
$$|m_1 - m_2| = 2$$
$$k_1 - k_2 = 0$$
$$\omega_1 - \omega_2 = 0$$

↓

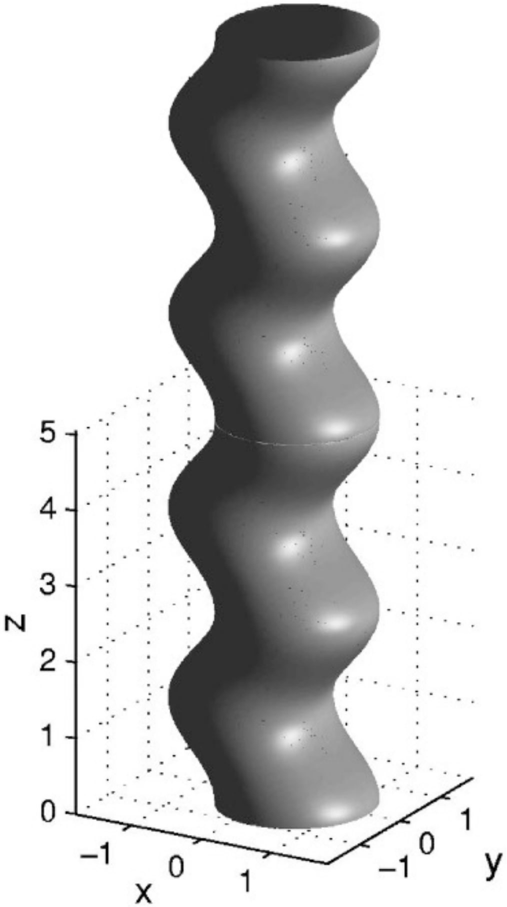
Amplification of perturbations
→ **Instability**

Short-wave instability mechanism

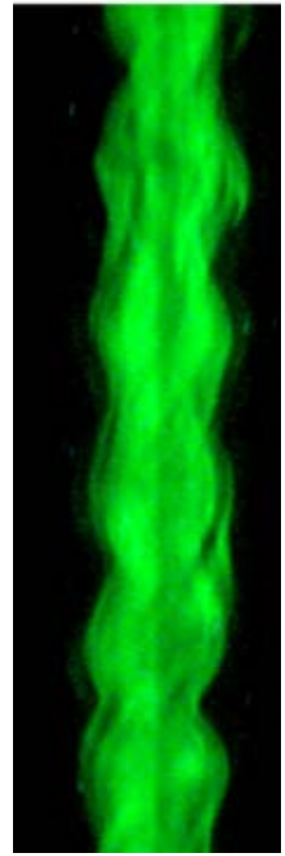
example: vortex in a strain - elliptic instability



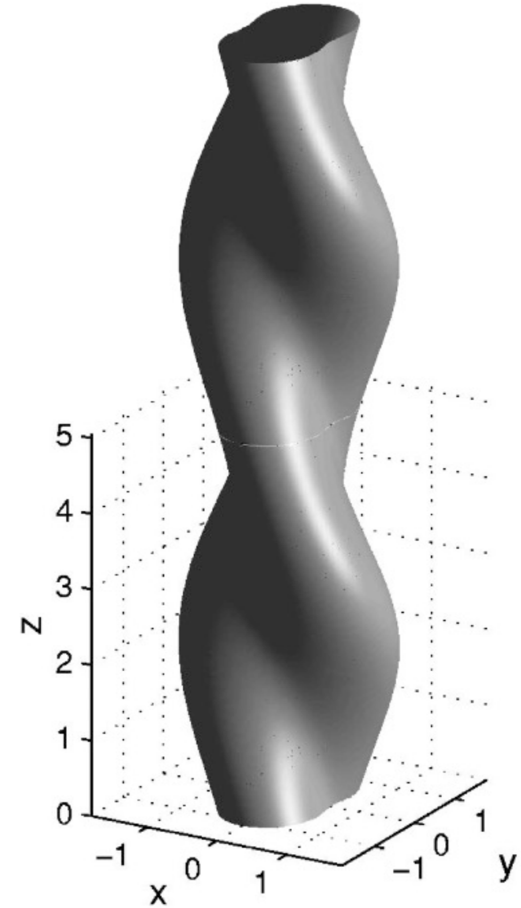
Experiment
vortex pair



Theory
m_1=-1, m_2=1



Experiment
vortex pair
with axial flow



Theory
m_1=2, m_2=0

Short-wave instability in the wake of a Boeing 747

31 March 2004 – flying from Marseille to Frankfurt

(courtesy of Charles Williamson, Cornell)

Short-wave instability in the wake of a Boeing 747

31 March 2004 – flying from Marseille to Frankfurt

(courtesy of Charles Williamson, Cornell)

Long-wave instability

Helical vortices

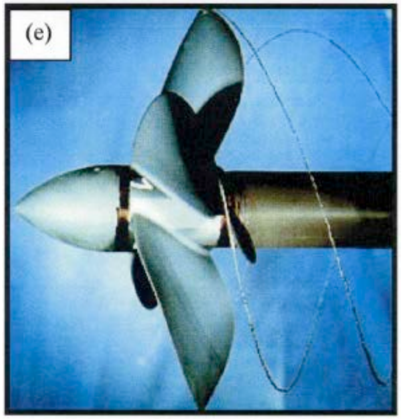
Pairing instability

Helical vortices

– Applications –

- helicopters
- propellers
- wind turbines

Hand *et al.* (2001)



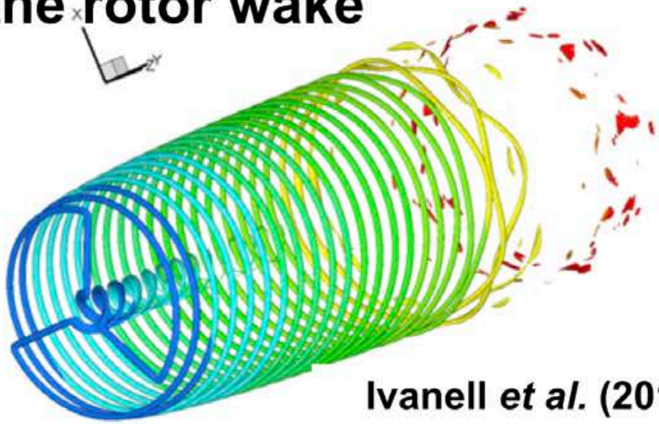
Senocak *et al.* (2002)

Helical vortices

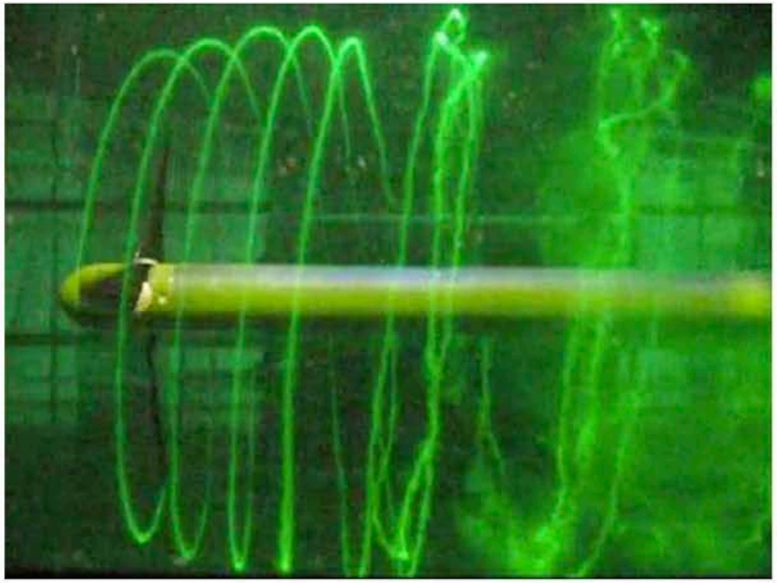
– Applications –

Wind turbines

- Downstream evolution of the rotor wake



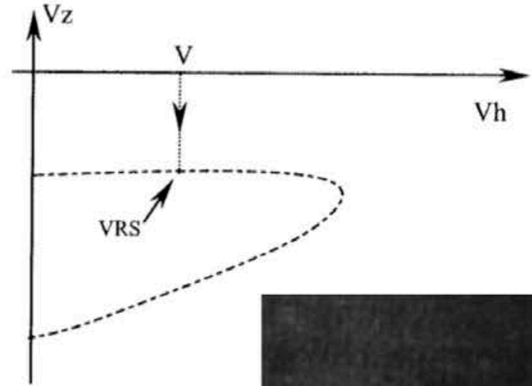
Ivanell *et al.* (2010)



Mikkelsen (2010, priv. comm.)

Helicopters

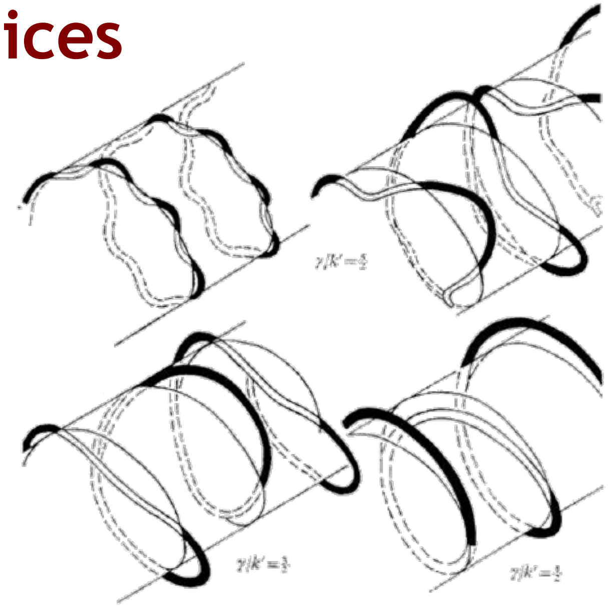
- transition from helical wake to Vortex Ring State (VRS)



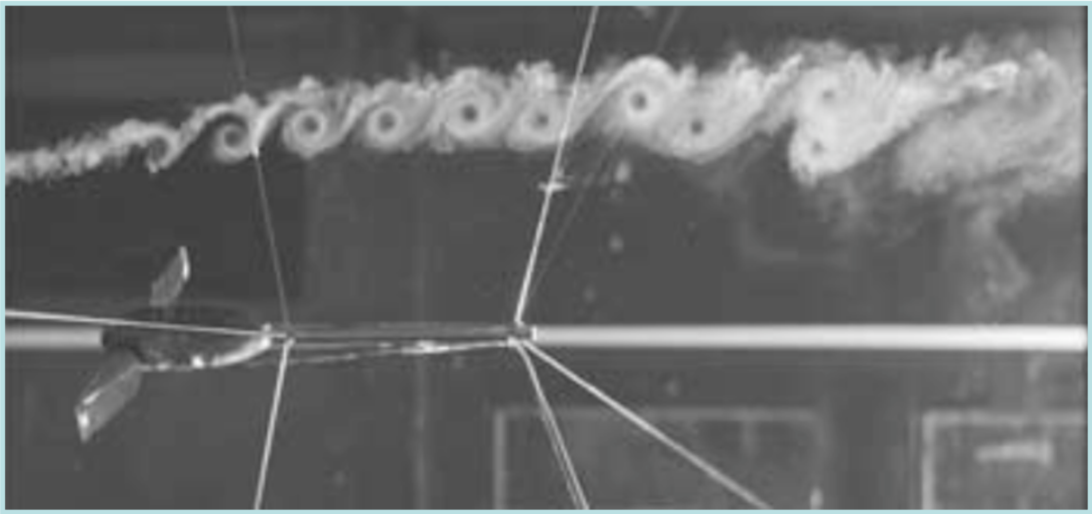
Meijer Drees & Hendal (1950)

Instabilities of helical vortices

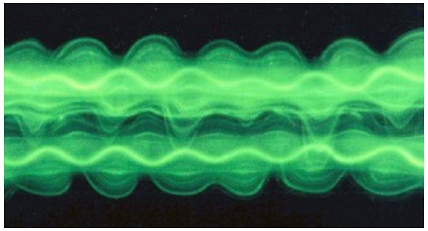
- Long-wave displacement instabilities (filaments)
- Short-wave core instabilities (elliptic, curvature)
- Swirling jet instability (vortex breakdown)



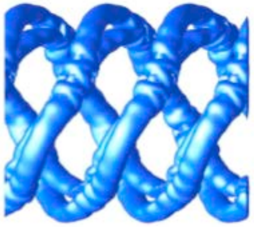
Widnall (1972)



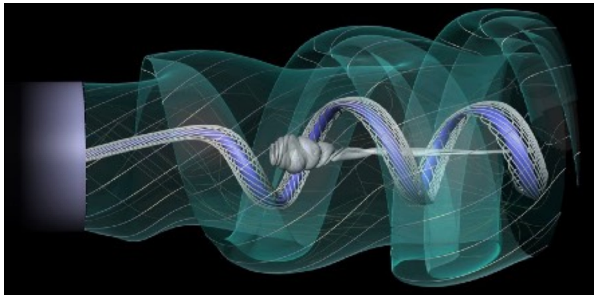
Alfredsson & Dahlberg (1979)



Leweke & Williamson (1998)

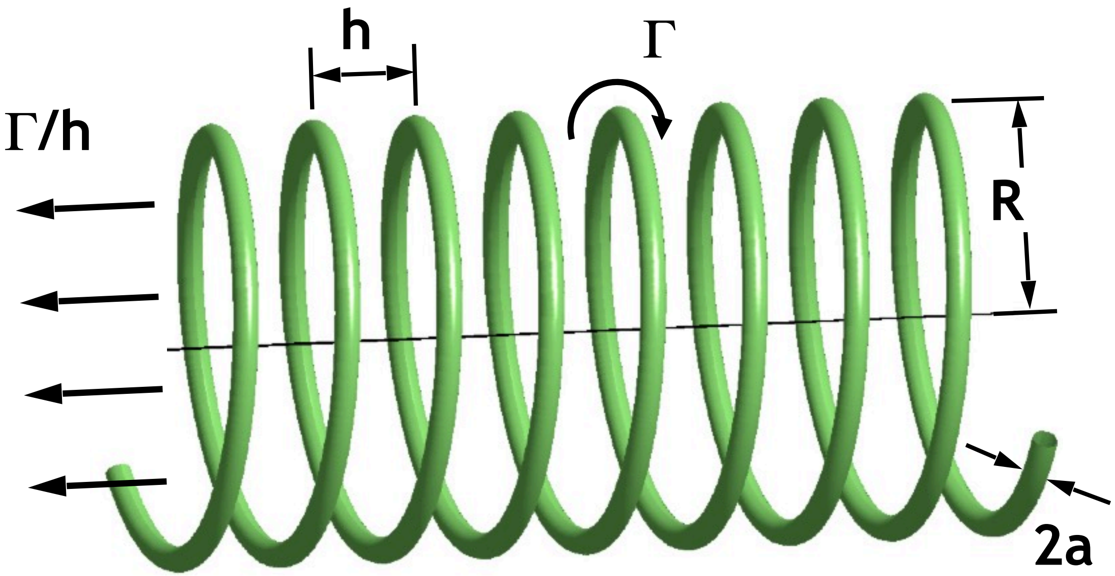


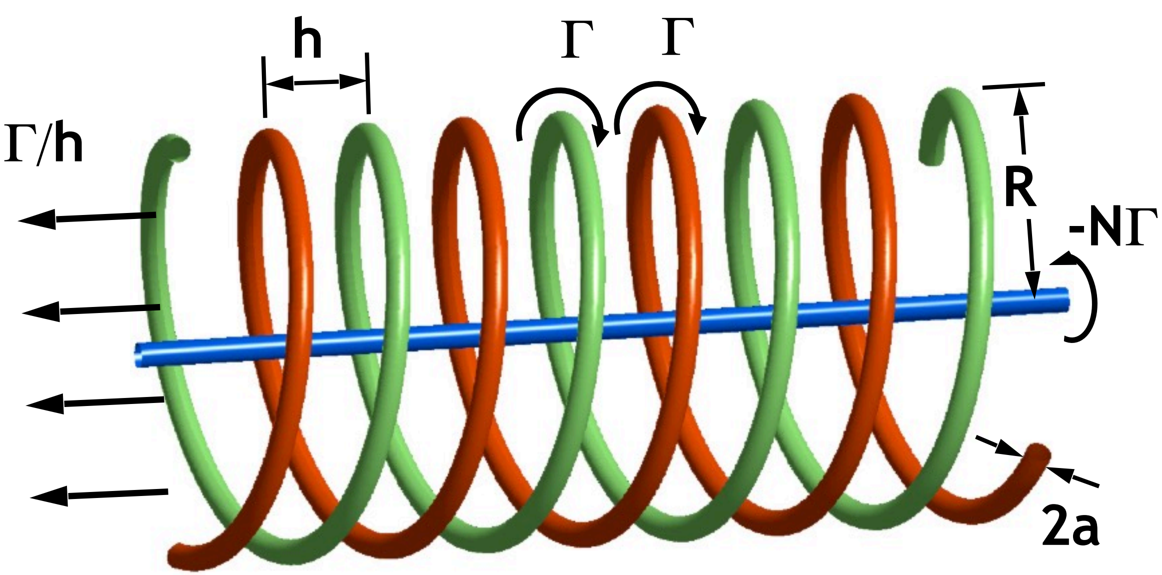
Walther *et al*. (2007)



Petz *et al*. (2011)

Configuration





infinite helical vortices

- circulation Γ
- radius R, pitch h $(\times N)$
- core radius a

parameters: R/a, h/a, $Re=\Gamma/\nu$
~40, ~20, ~10^4

<u>characteristic scales</u>

- advection velocity : ~$\Gamma/2h$
- length scale : h
- time scale : $2h^2/\Gamma$

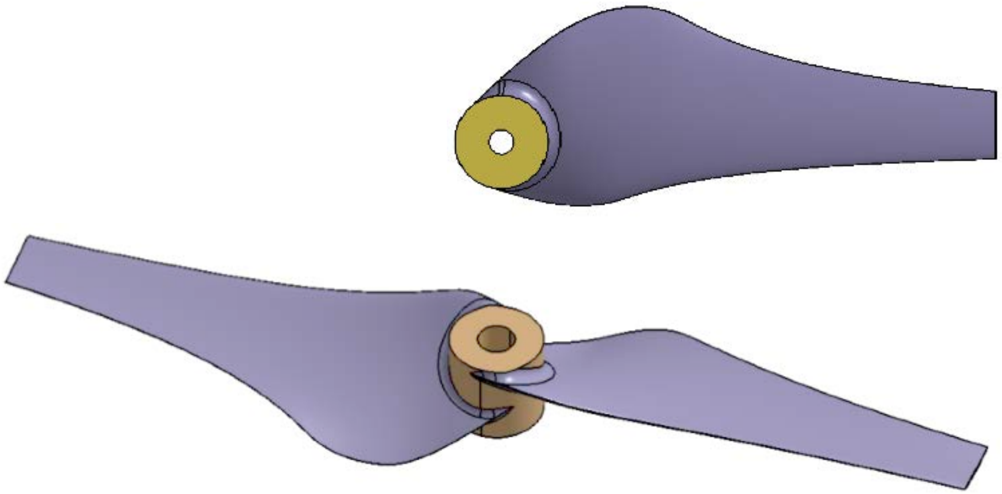
➔ $\sigma^* = \sigma \cdot (2h^2/\Gamma)$

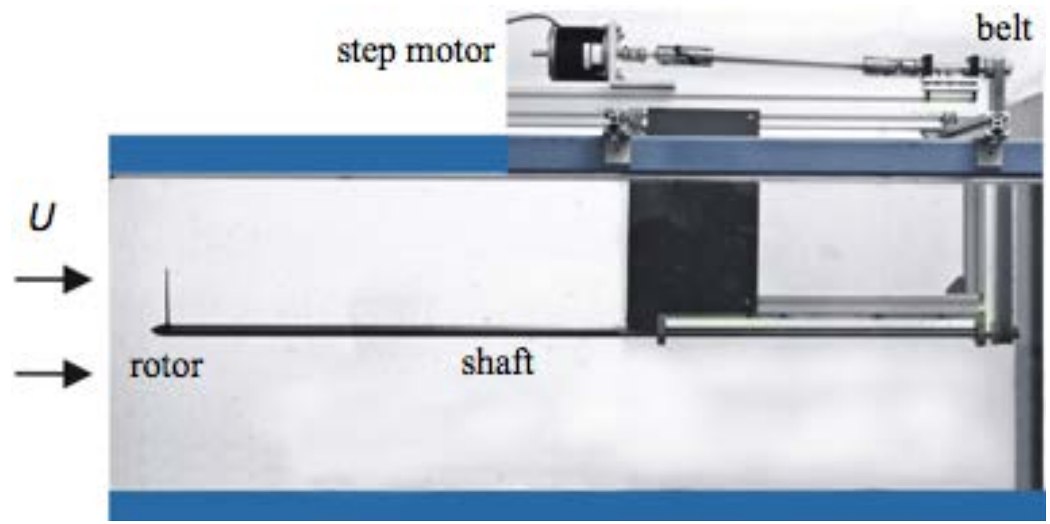
Experimental study

Water channel:
- recirculating, free surface
- test section 37 cm × 50 cm × 150 cm
- free stream velocity *U* = 35-45 cm/s

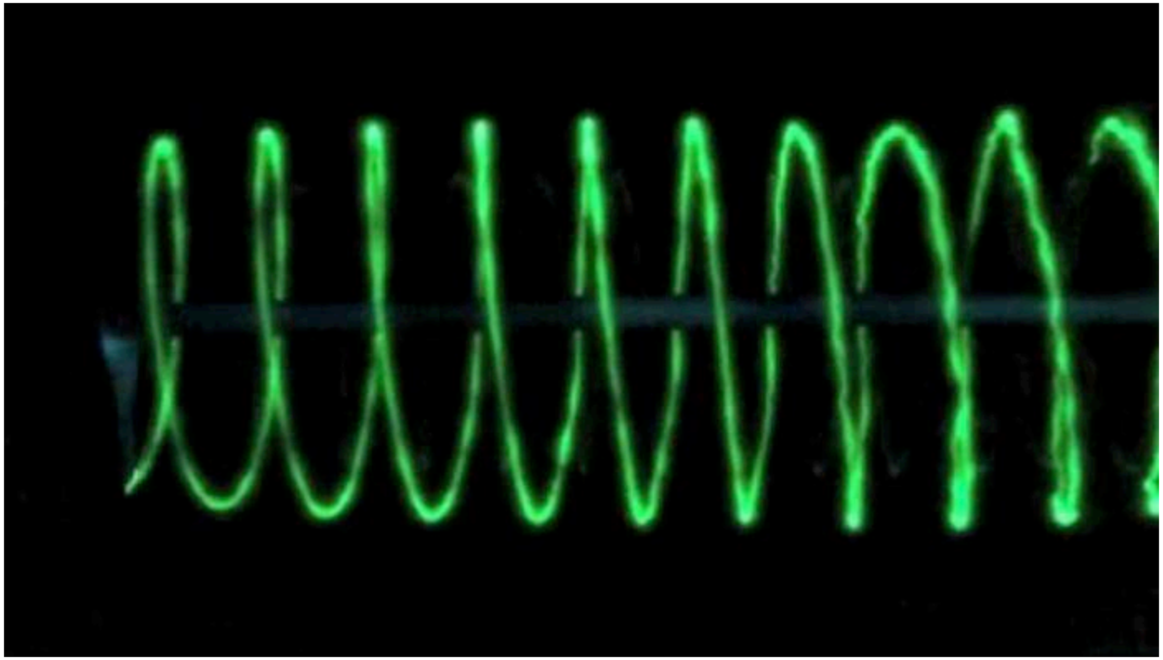
Rotors:
- one or two blades
- low-Re airfoil (A18 – M.S. Selig, UIUC)
- tip chord *c* = 11 mm - Re_c= 30000
- radius: 80 mm
- constant circulation
- Ω = 3-6 rps   tip speed ratio λ = 5-9

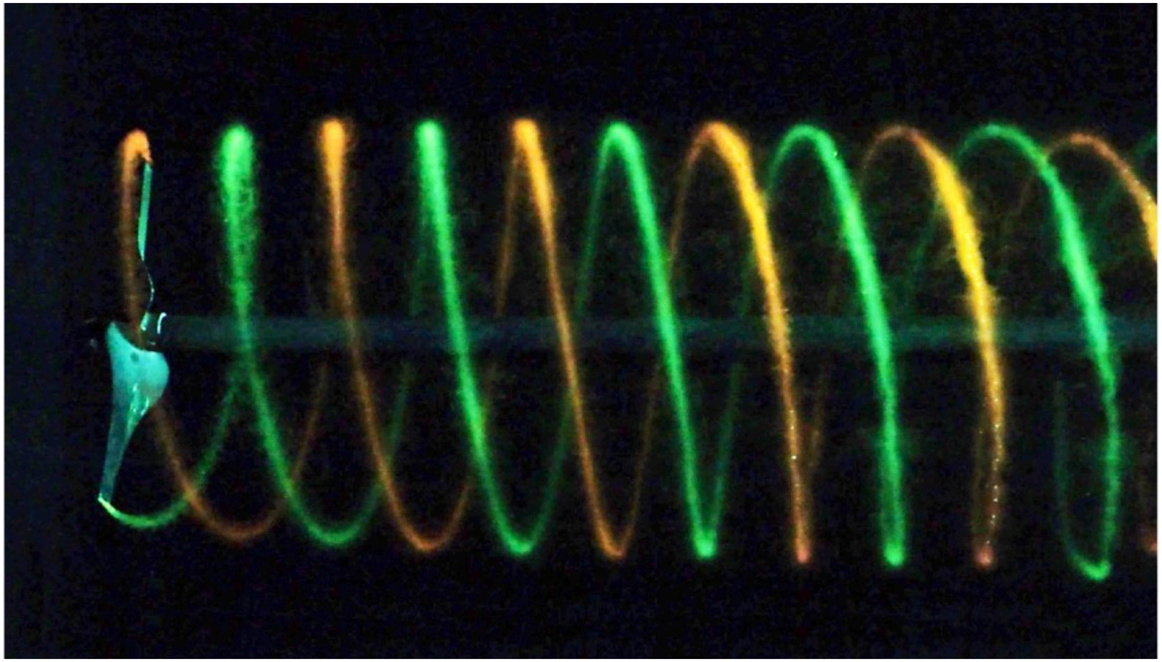
Methods:
- dye visualisations
- Particle Image Velocimetry (2C & 3C)
- high-speed video & PIV





Experimental study





Long-wave instability - theoretical results

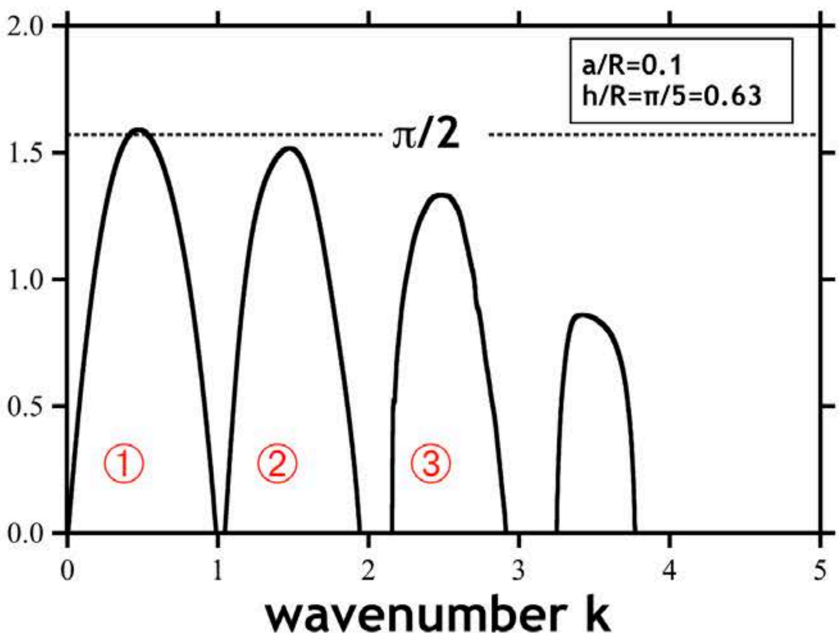
(Widnall 1972, Gupta & Loewy 1974, Fukumoto & Miyazaki 1991)

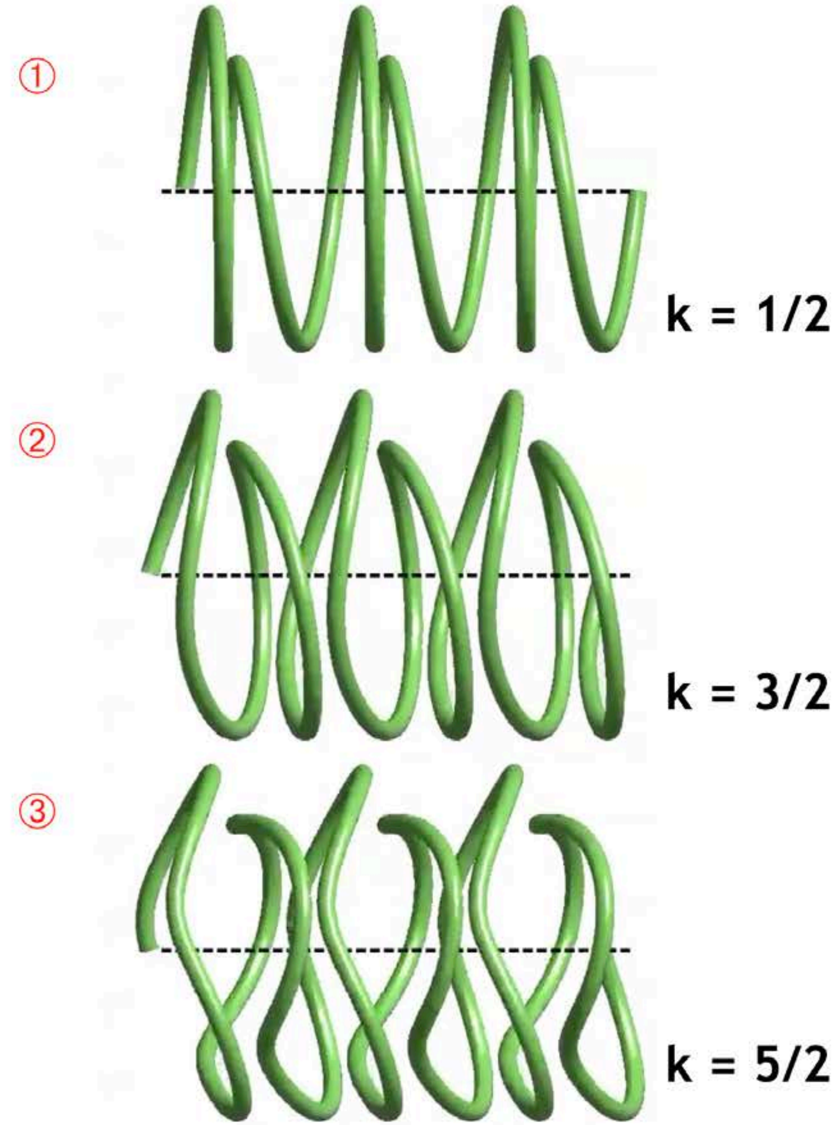
single helical vortex

$\boxed{\sigma^* = f(a/R,\ h/R,\ k)}$

number of waves in one turn

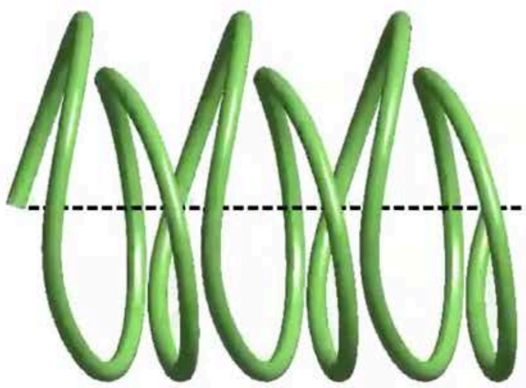
$\sigma^* = \sigma \cdot (2h^2/\Gamma)$





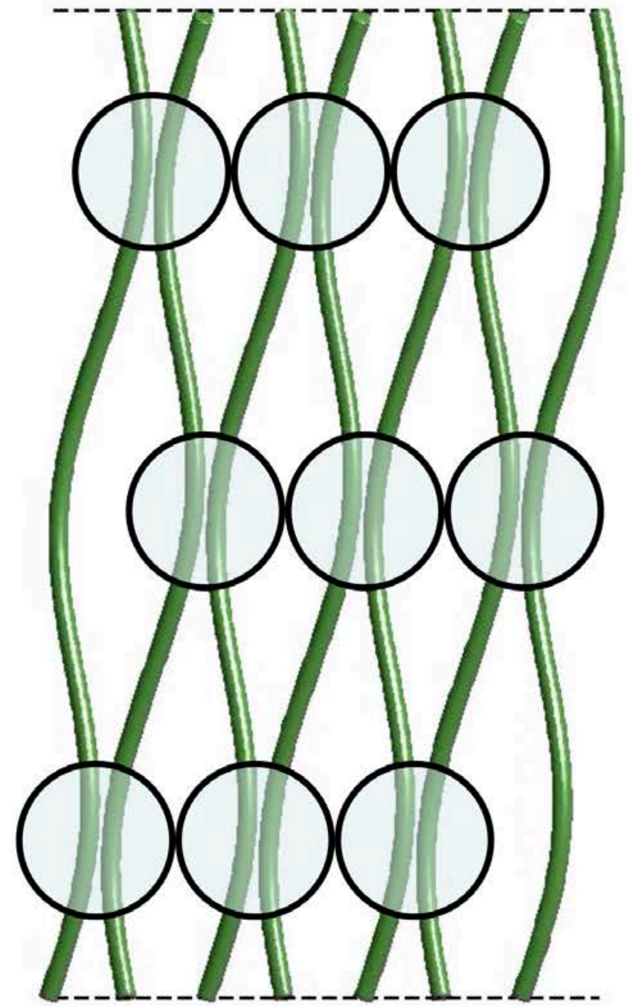
Long-wave instability - theoretical results

side view



k = 3/2

developed plan view



k = m/n

m locations of local pairing

in n helix turns

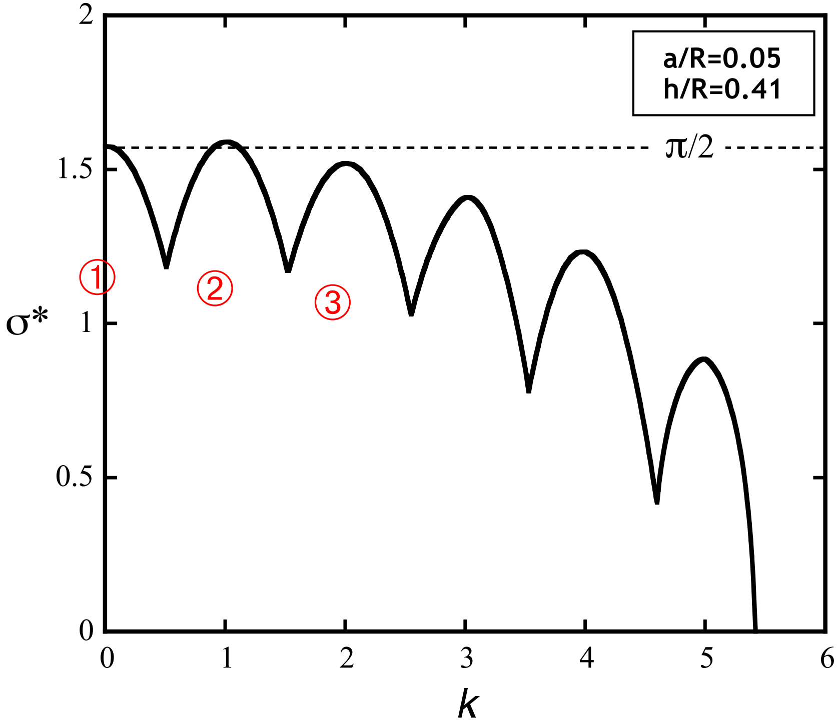
Long-wave instability - theoretical results

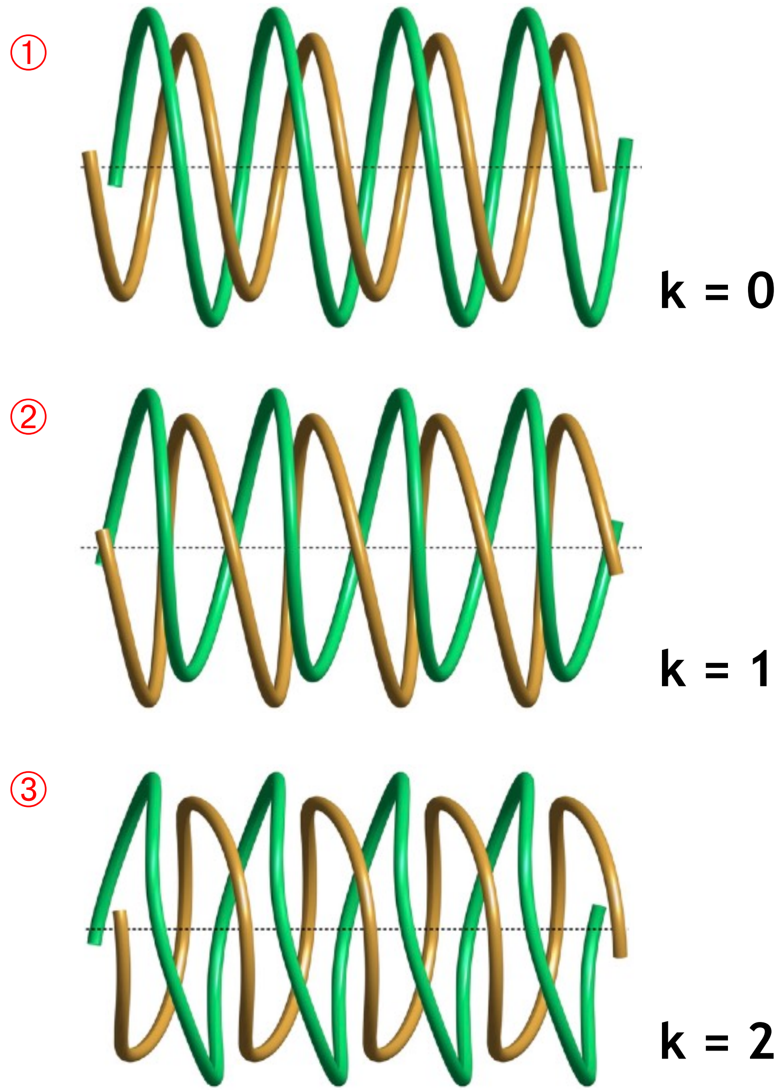
(Gupta & Loewy 1974, Okulov & Sørensen 2007)

two interlaced helical vortices

$\sigma^* = f(a/R,\ h/R,\ k)$

number of waves in one turn

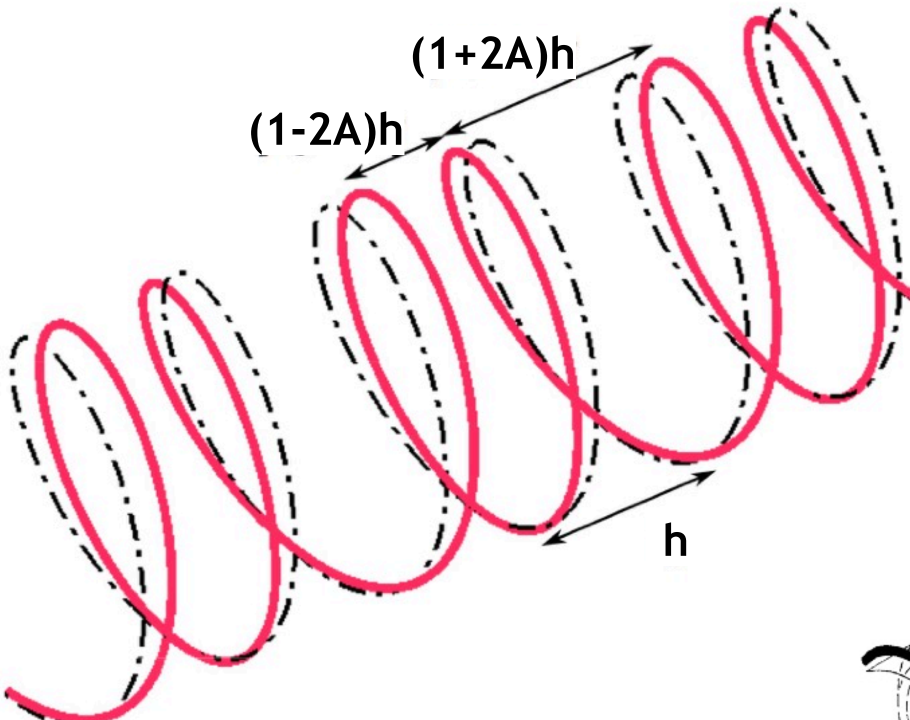


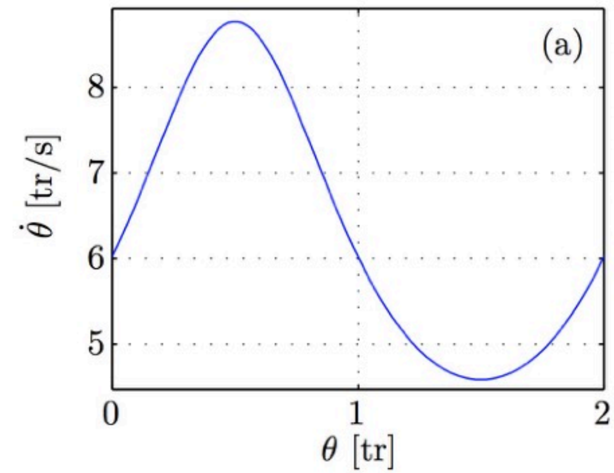


Perturbed flow - <u>single</u> helix

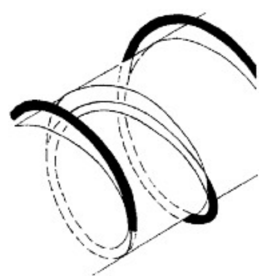
Perturbation of the helix geometry: $\boxed{z/h \ = \ \theta/2\pi \ + \ A \cos(k\theta)}$

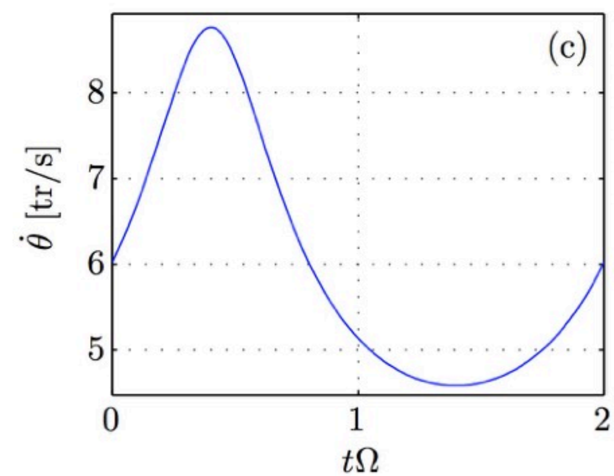
Example: k = 1/2 and A = 0.15 (for R/h=2)





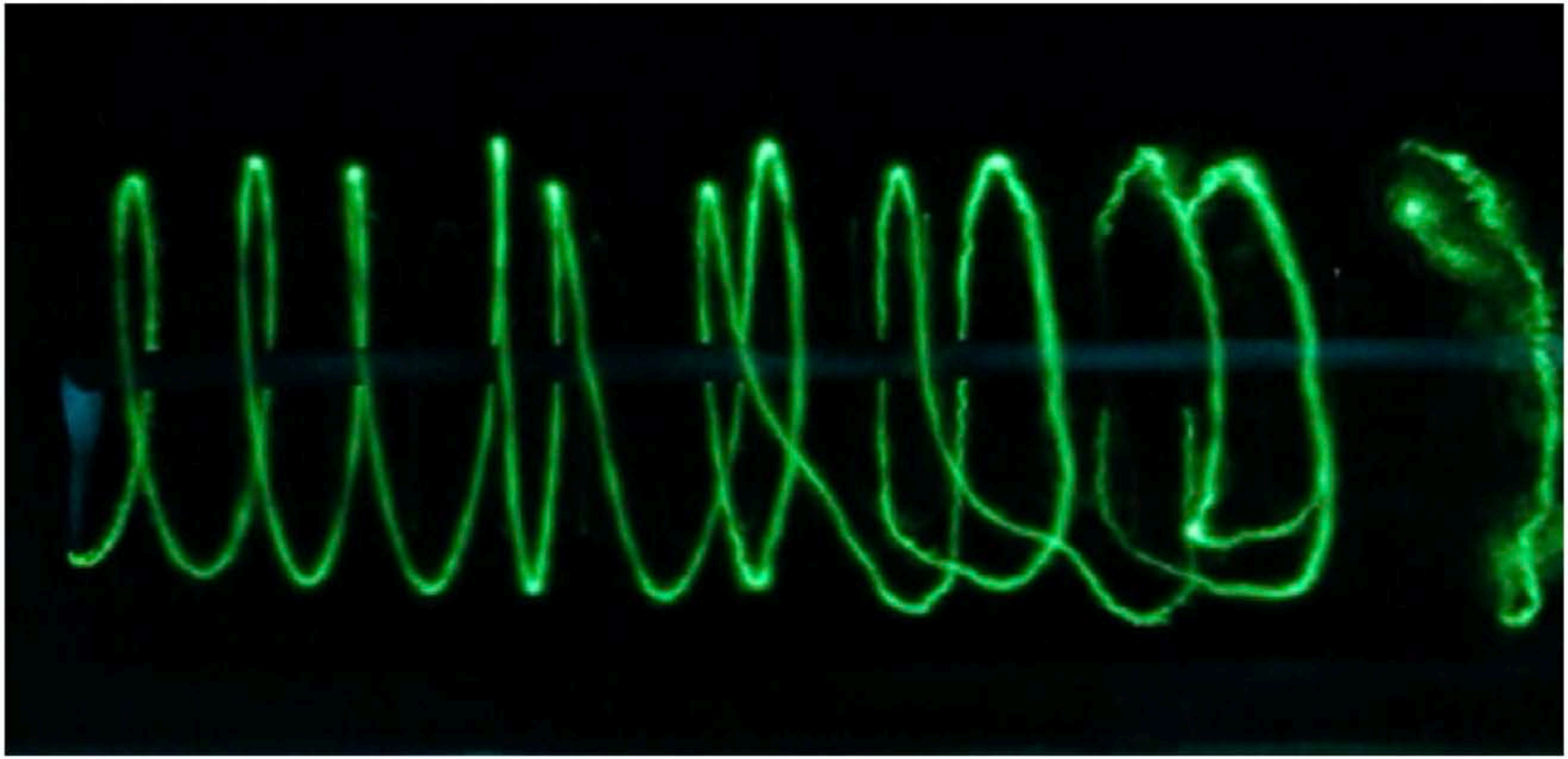
→ Widnall mode with k = 1/2





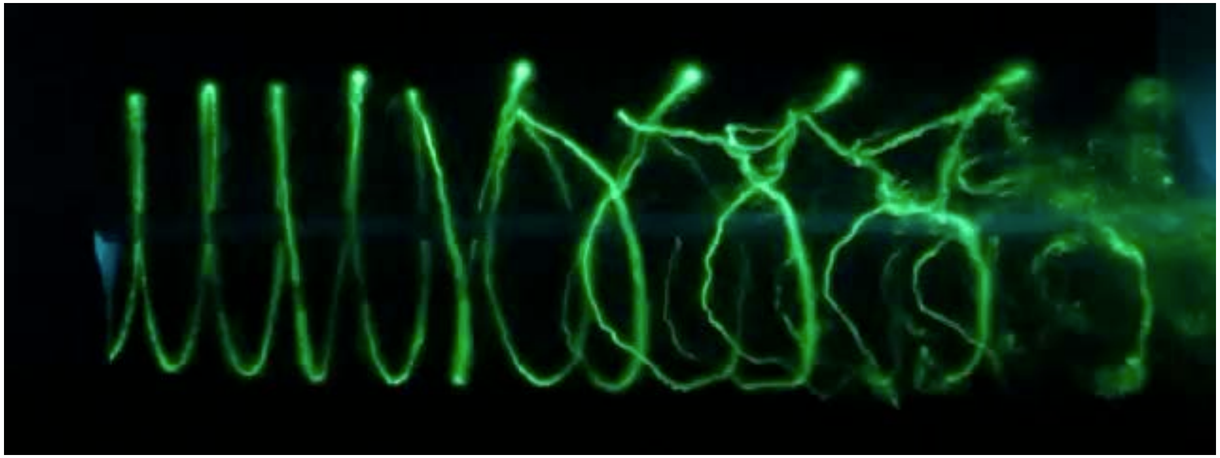
Perturbed flow – <u>single</u> helix

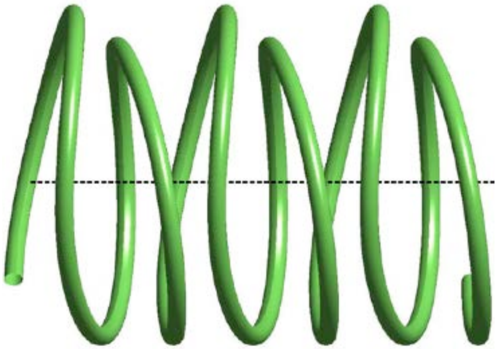
k = 1/2 - A = 0.05



Perturbed flow - single helix

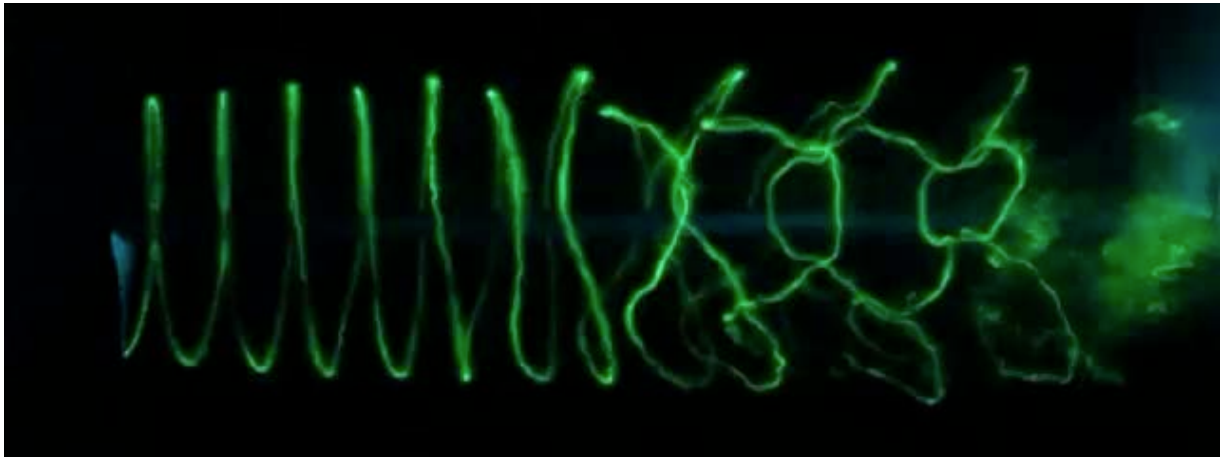
k = 3/2 - A = 0.03

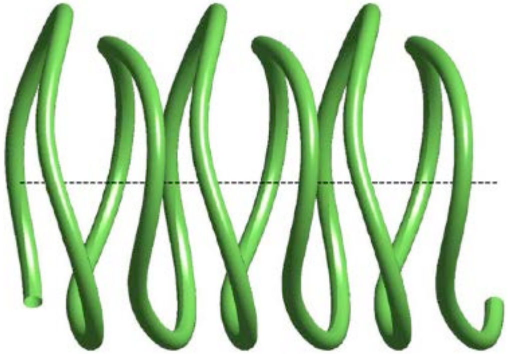




k = 3/2

k = 5/2 - A = 0.01

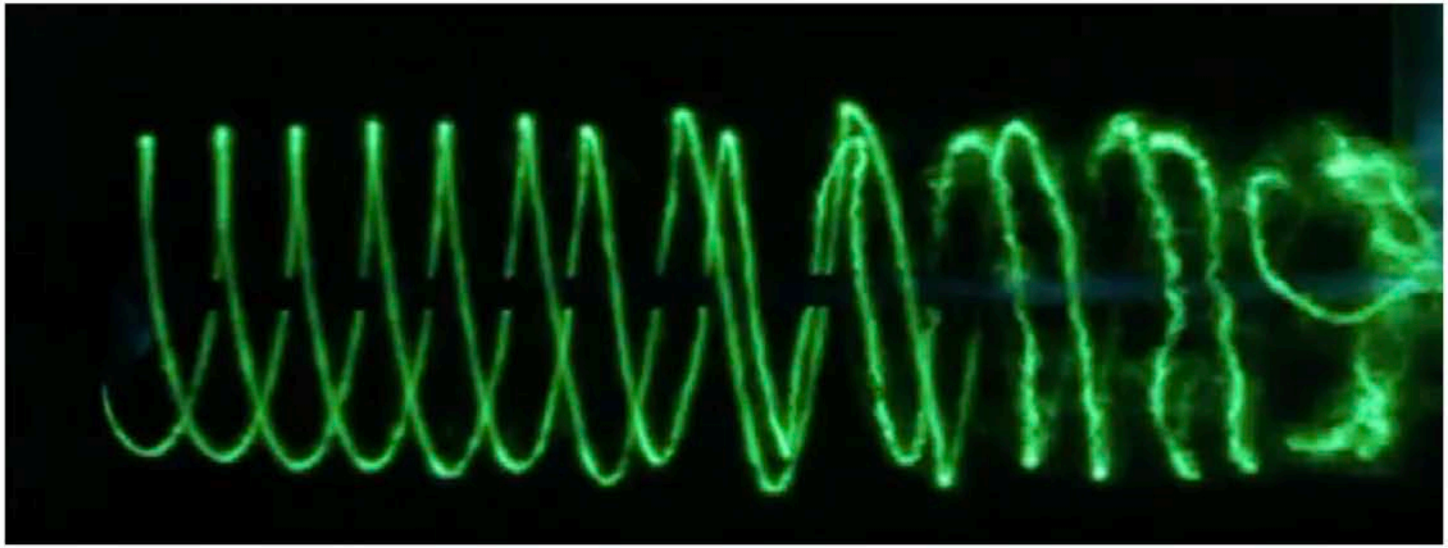




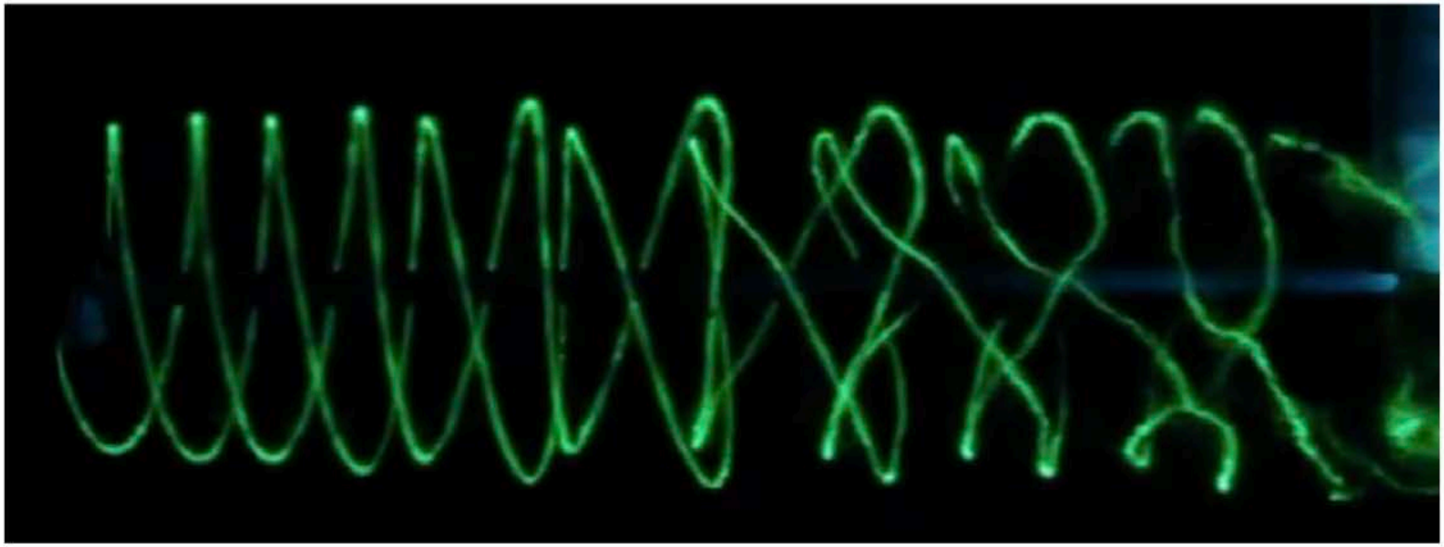
k = 5/2

Perturbed flow – double helix

uniform pairing
k = 0

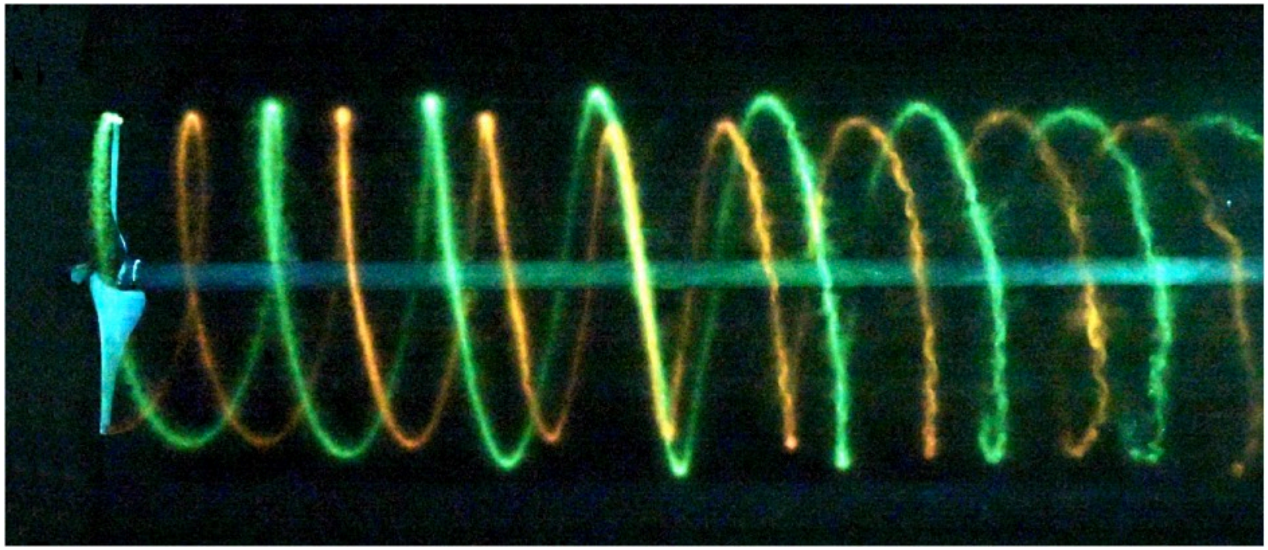


local pairing
k = 1

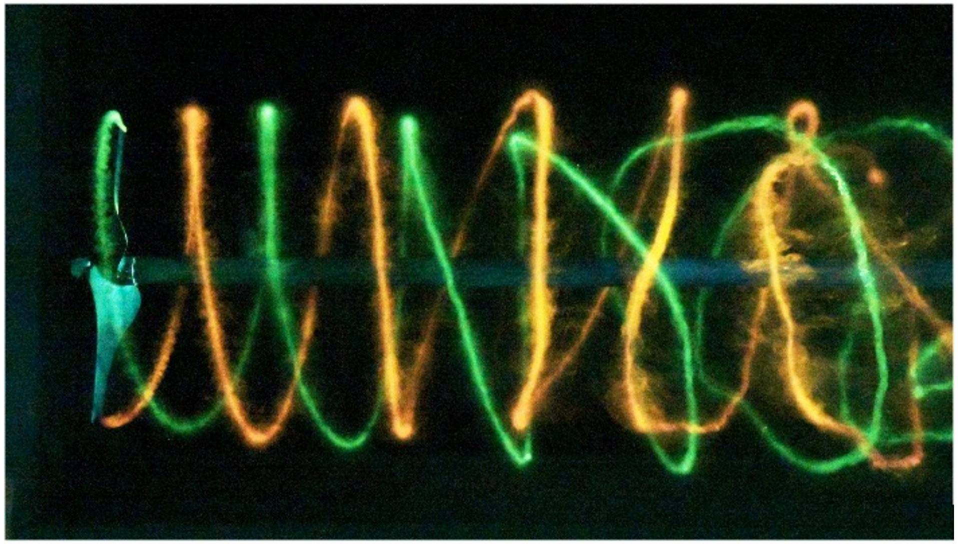


Perturbed flow - double helix

uniform pairing
k = 0

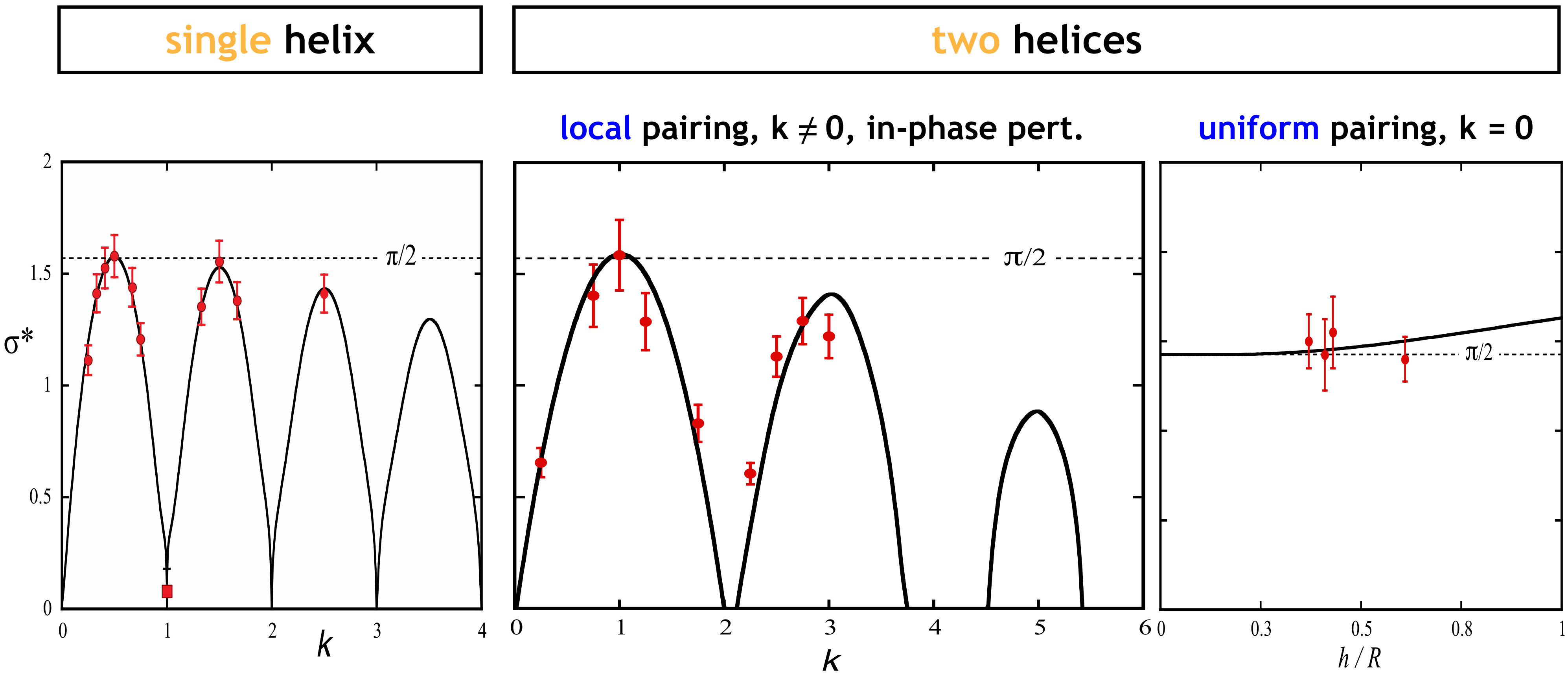


local pairing
k = 1



Growth rate of long-wave instabilities

Experiment vs. theory (Gupta & Loewy 1974)

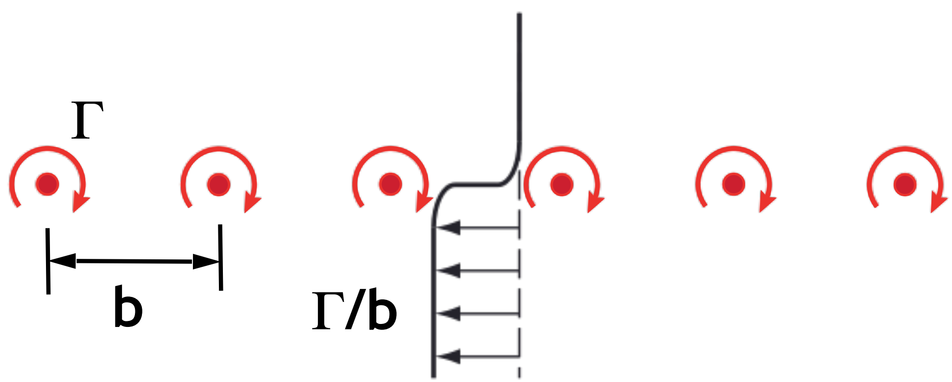


Pairing instability

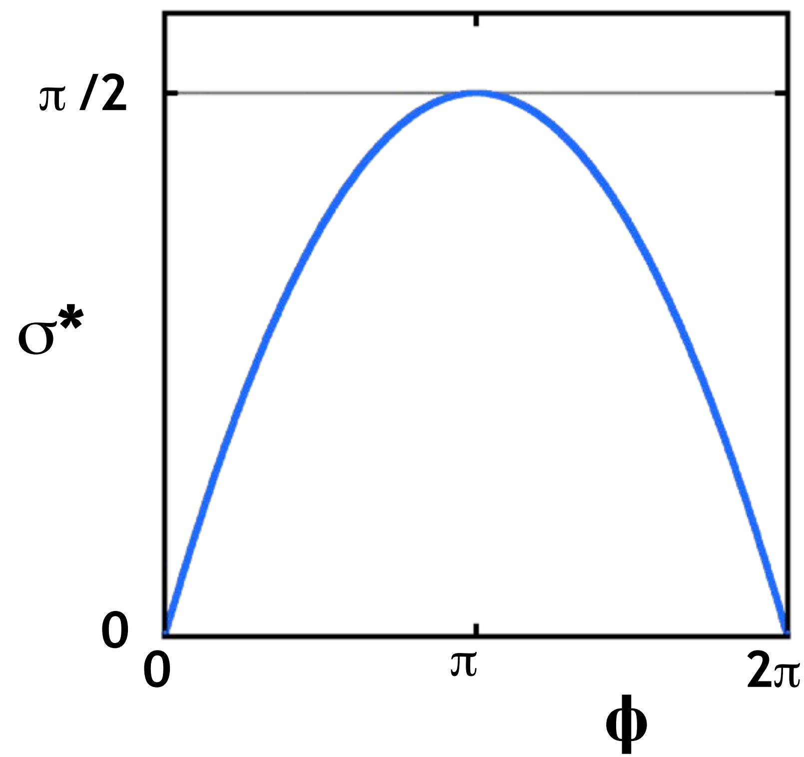
2 dimensions (von Kármán & Rubach 1912, Lamb 1932)

$\sigma^* = \pi/2$ for $\phi = \pi$

infinite row of point vortices



$$\sigma^* = \sigma \cdot (2b^2/\Gamma)$$



Pairing instability

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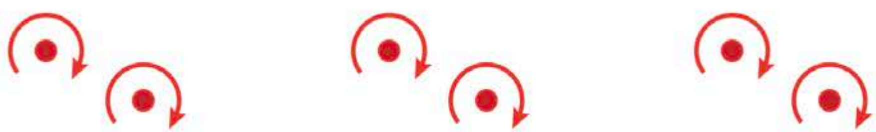
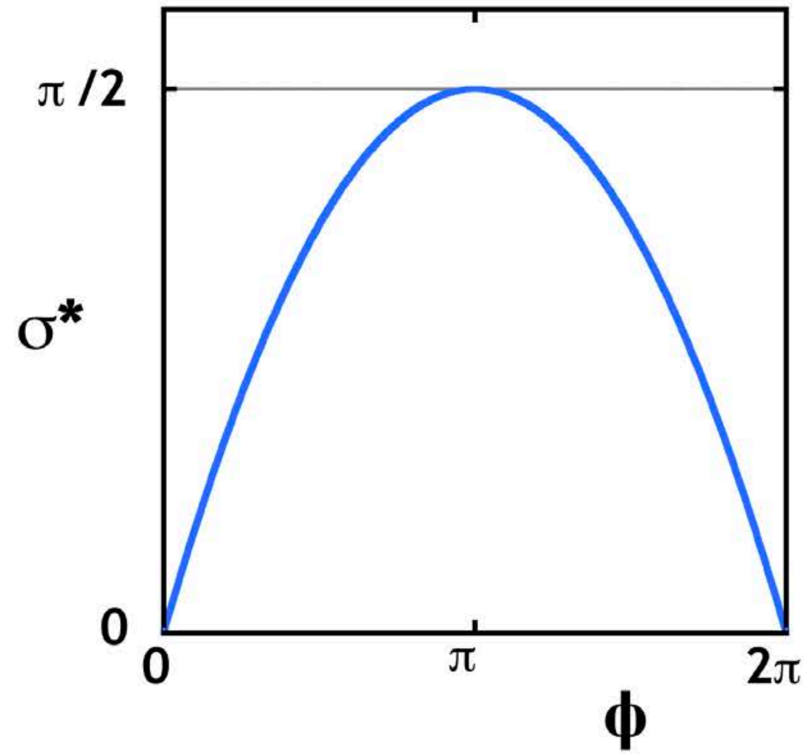




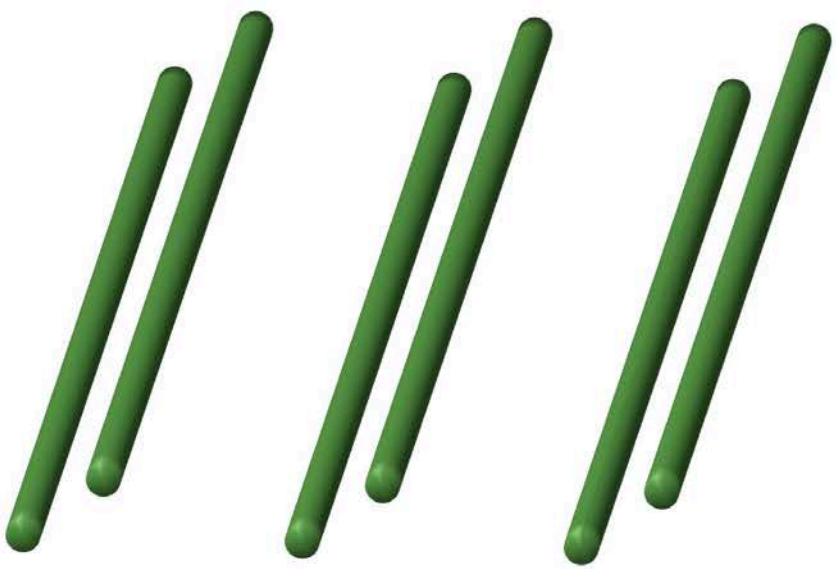
$$\boxed{\sigma^* = \sigma \cdot (2b^2/\Gamma)}$$

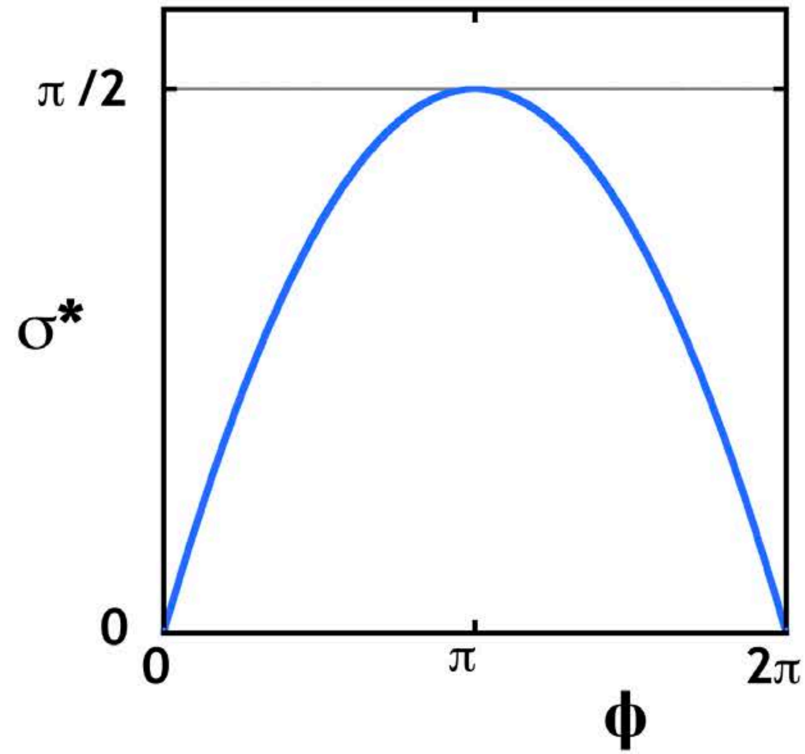
Pairing instability

2 dimensions (von Kármán & Rubach 1912, Lamb 1932)

$\sigma^* = \pi/2$ for $\phi = \pi$

infinite array of straight vortices





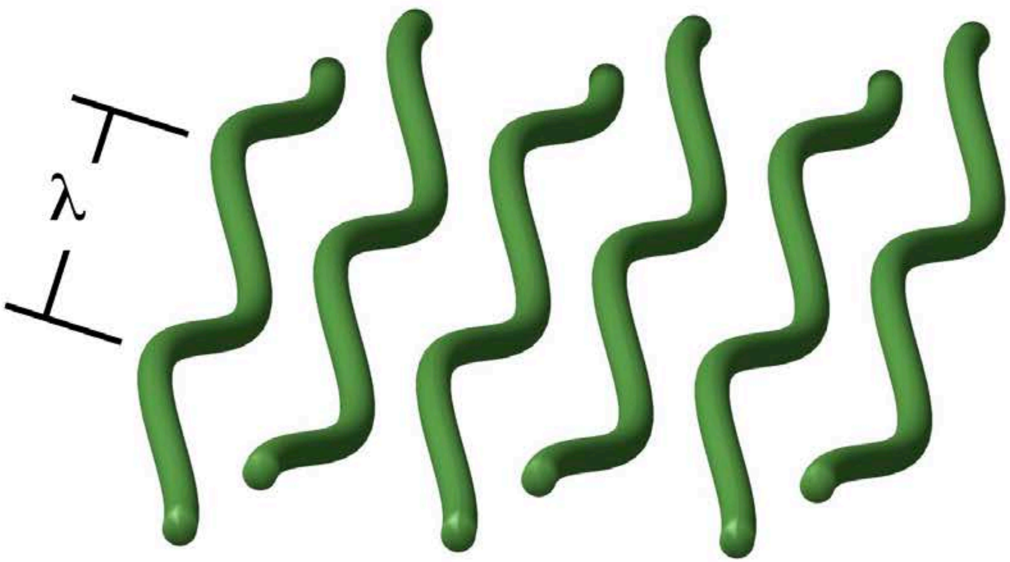
Pairing instability

3 dimensions (Robinson & Saffman1982)

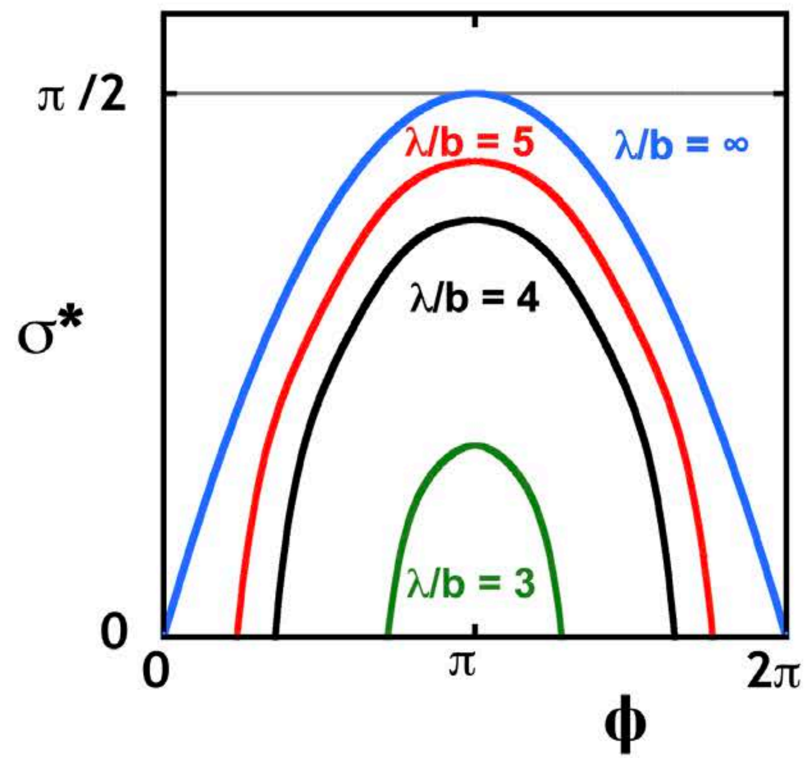
```matlab
% Robinson & Saffman (1982)
% single row

a_sur_l = 0.1;     % a/l
l_sur_lambda=1/4;
i=0;
for phi=0:2*pi/100:2*pi;
    i=i+1;
    phi_plot(i)=phi;
    kl = 2*pi*l_sur_lambda;
    eta= kl^2/2*(log(1/(pi*a_sur_l*l_sur_lambda)) ...
        -0.5772+0.25);
    A  = pi^2/3;
    At = A;
    for p=1:50;
        A  = A  - 2*psi_ind(p*kl)/p^2*cos(p*phi);
        At = At - 2*xsi_ind(p*kl)/p^2*cos(p*phi);
    end
    sigma(i) = 1/pi*real(sqrt((A-eta)*(At+eta)));
end
plot(phi_plot,sigma)
```

infinite array of straight vortices

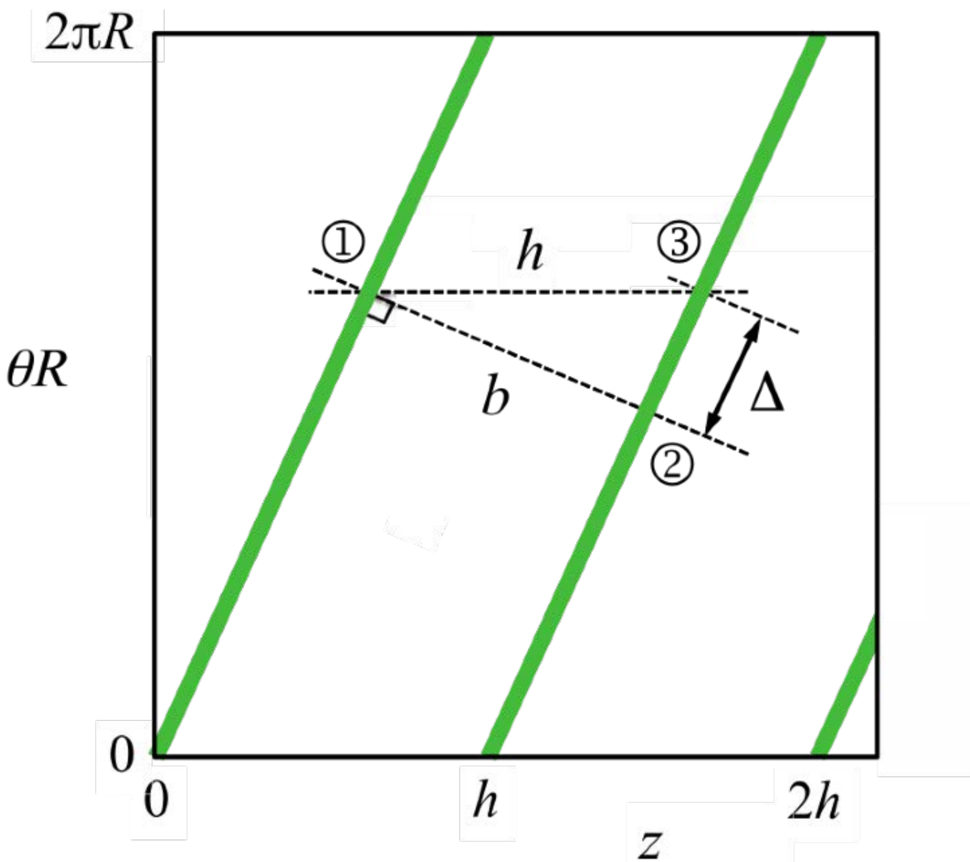


a / b = 0.1



Pairing instability

prediction for helical geometry



Helical vortex

R, h, k, Γ, a

↓

Equivalent array of straight vortices

$b = f(h,R)$ $\quad \Gamma$

$\phi = f(k,R,h)$ $\quad a$

$\lambda = f(k,R,h)$

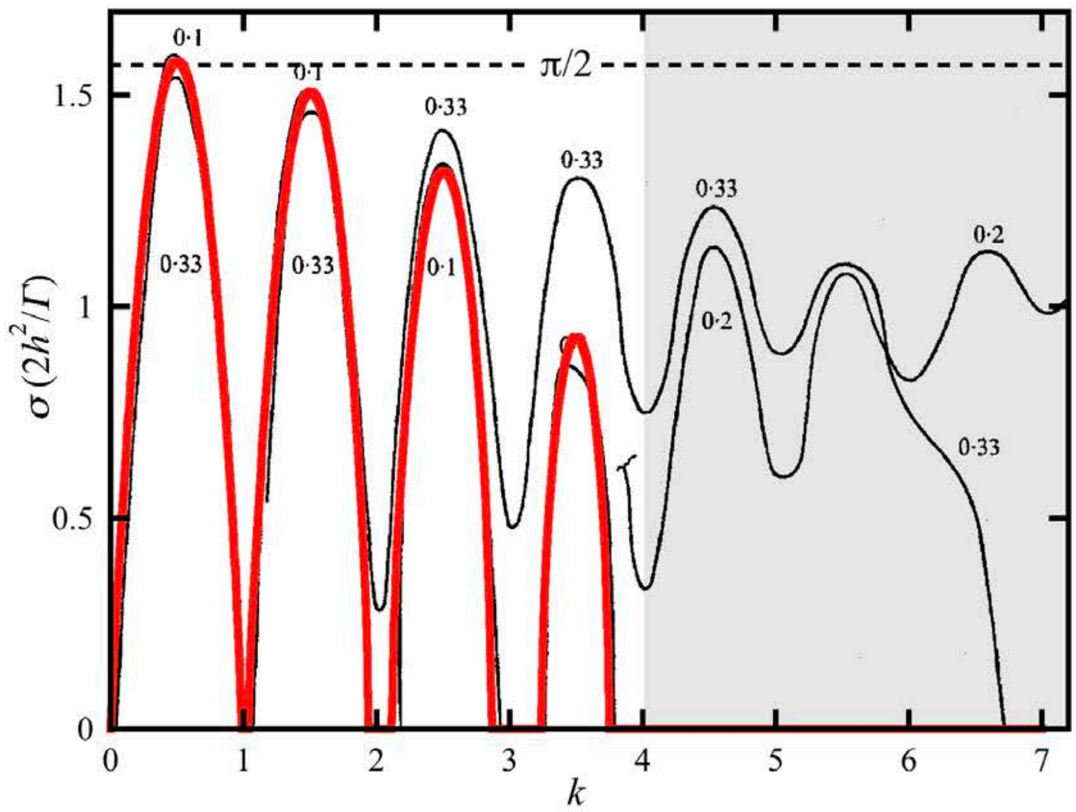
↓

$\sigma \cdot (2h^2/\Gamma)$

Long-wave instability – theoretical results

Growth rate for h/R = π/5 = 0.63

$a / R = 0.1$



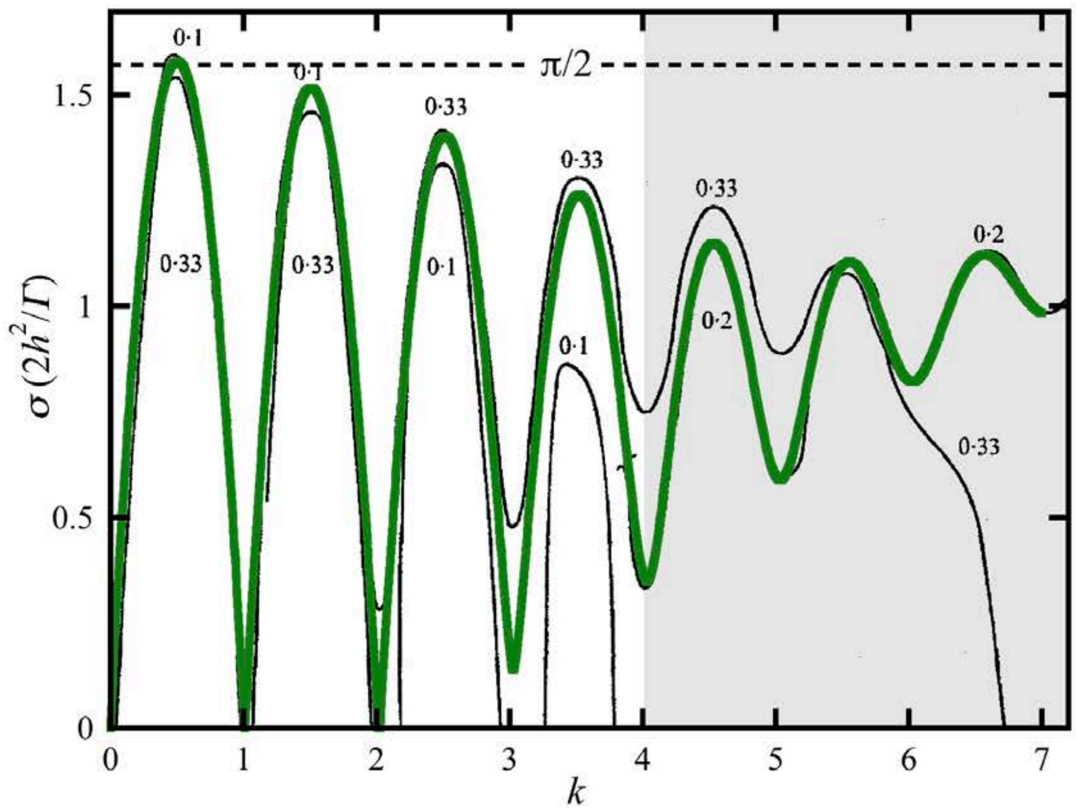
Robinson & Saffman (1982)

applied to helical geometry

Long-wave instability – theoretical results

Growth rate for h/R = π/5 = 0.63

$a / R = 0.2$



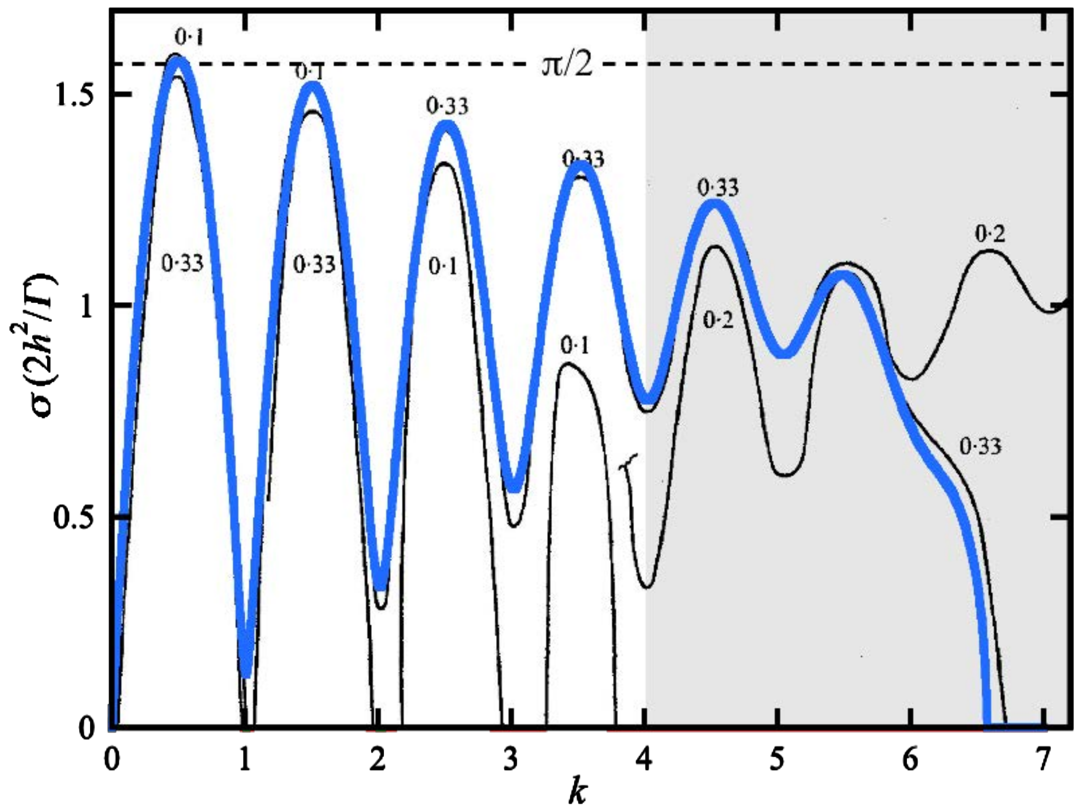
Robinson & Saffman (1982)

applied to helical geometry

Long-wave instability – theoretical results

Growth rate for h/R = π/5 = 0.63

$a / R = 0.33$

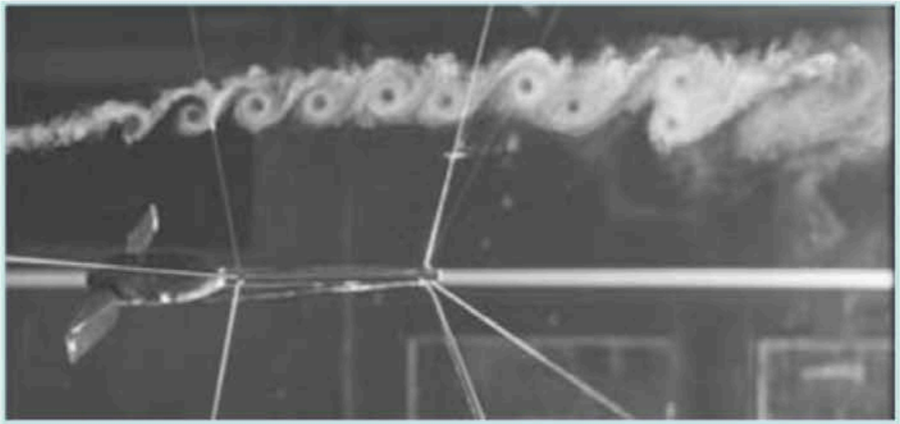


Robinson & Saffman (1982)

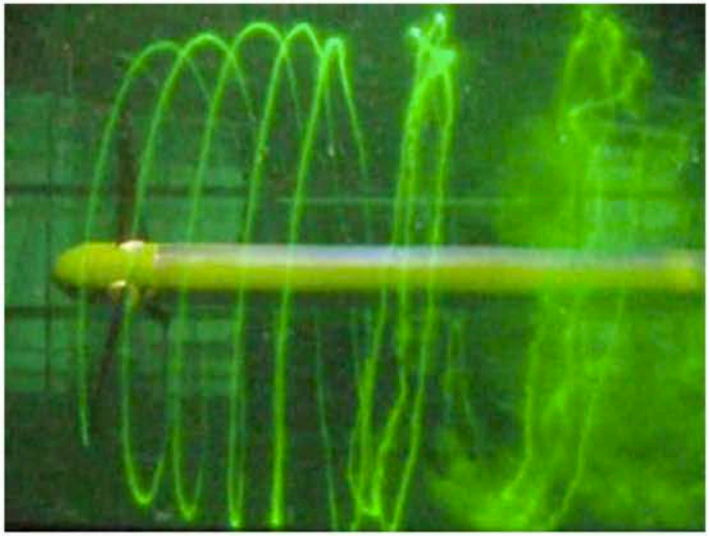
applied to helical geometry

Experiments with rotors of more than one blade

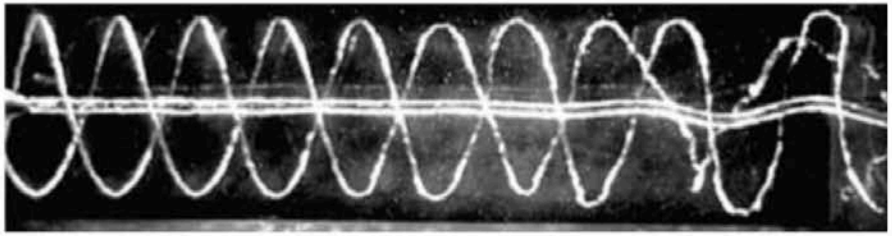
Wind turbine in air (Alfredsson & Dahlberg,1979)



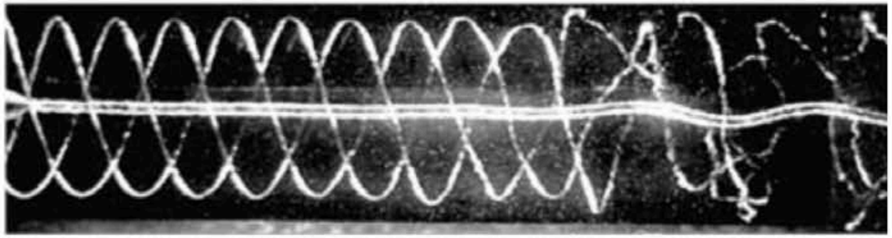
Wind turbine in water (Mikkelsen,2010)



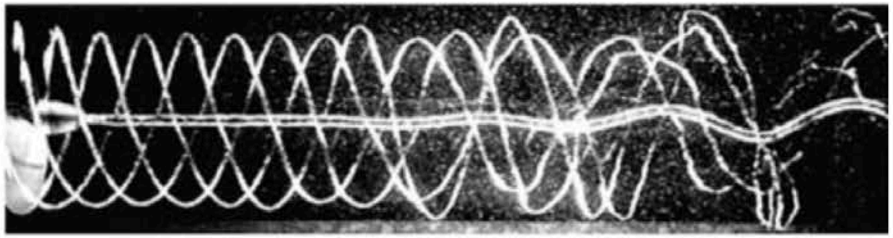
Propeller in water (Felli *et al.*, 2011)



2 blades



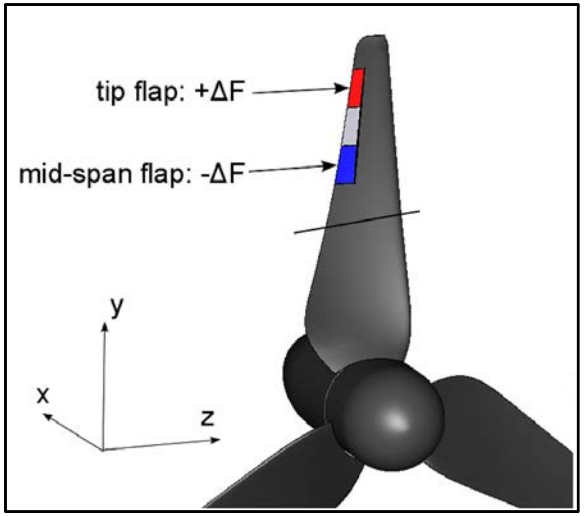
3 blades

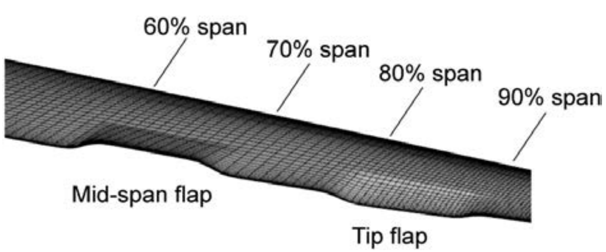


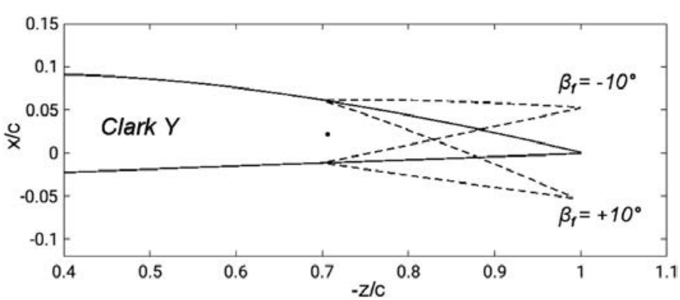
4 blades

Application to wind turbines

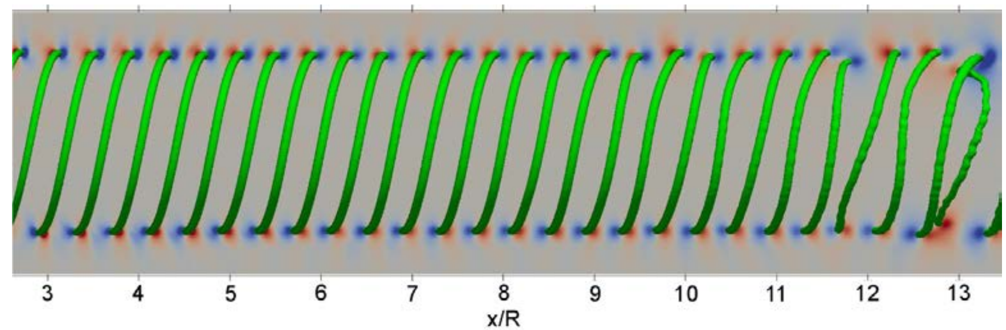
Huang *et al*. (2019)



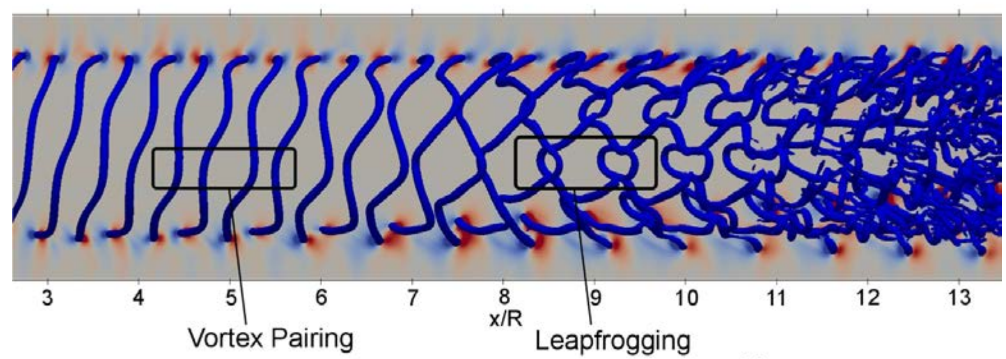




without flap oscillation



with flap oscillation

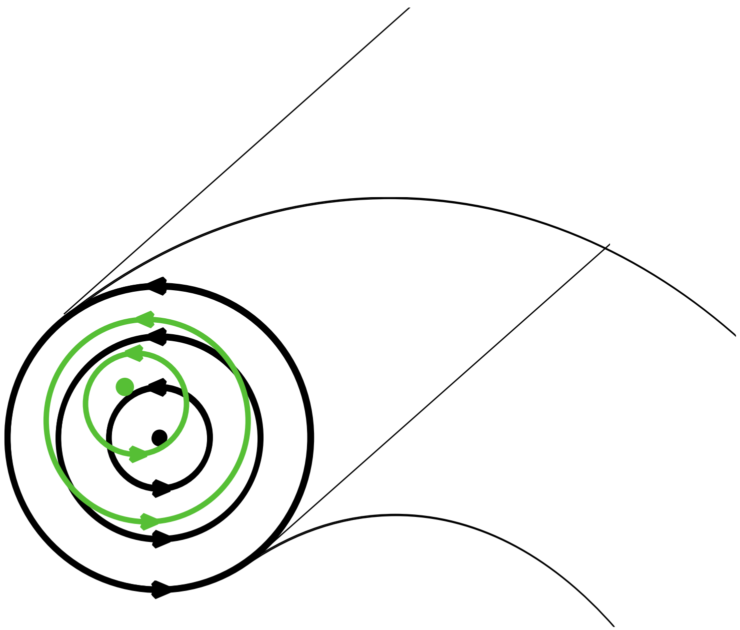


Short-wave instability
Helical vortices

Curvature instability

Short-wave instability mechanism

curved vortex - curvature instability



curvature perturbation
(m = 1, k = 0)

Kelvin modes
$u(r) \cdot \exp[i(kz + m\theta + \omega t)]$

↓

Resonance between two modes (m_1, k_1, ω_1) and (m_2, k_2, ω_2) and a **curvature perturbation** if

$$|m_1 - m_2| = 1$$
$$k_1 - k_2 = 0$$
$$\omega_1 - \omega_2 = 0$$

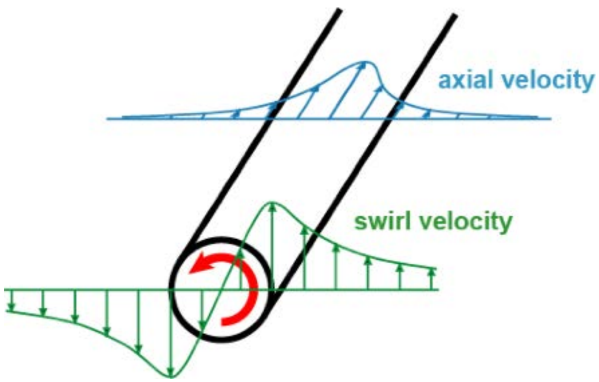
↓

Amplification of perturbations
→ **Instability**

Short-wave instability

Theoretical prediction for experimental case

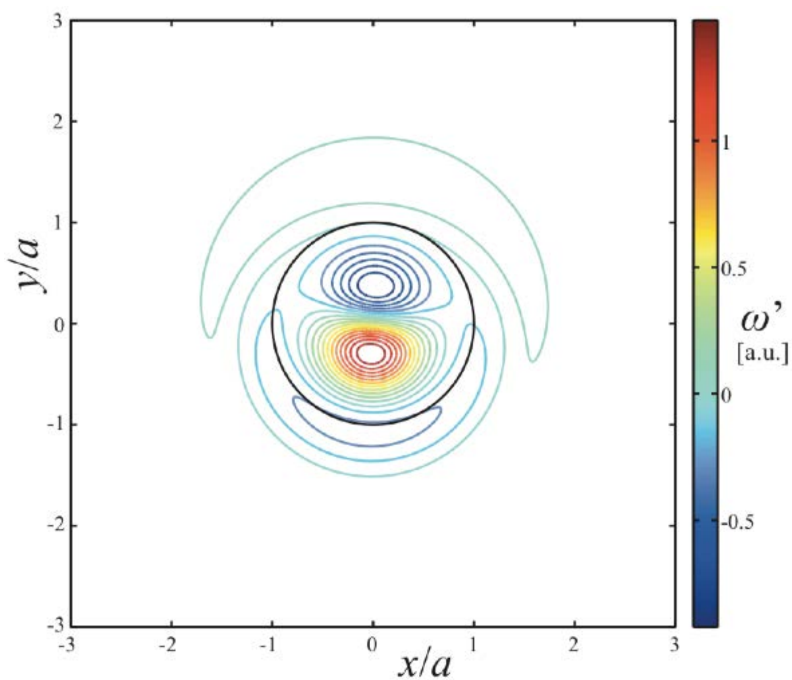
(Blanco-Rodríguez & Le Dizès, 2017)



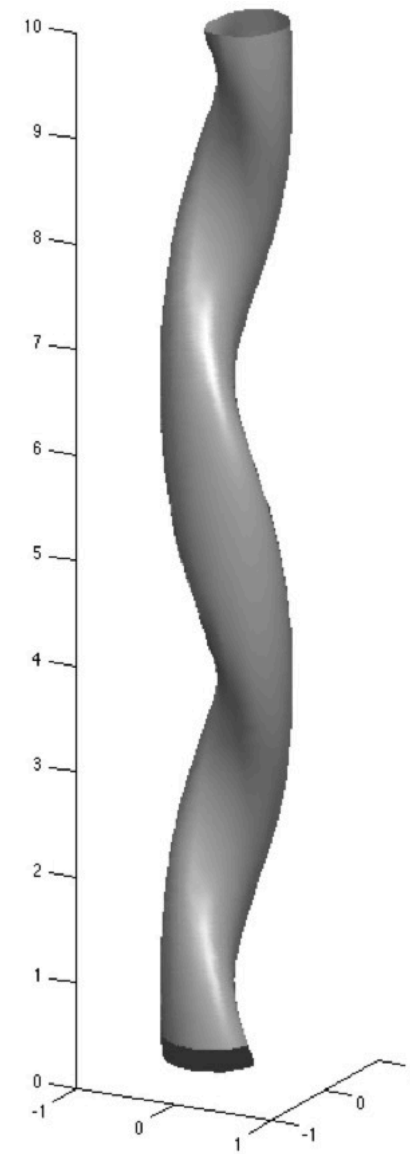
<u>Gaussian</u> vortex with <u>axial flow</u>

h/R = 0.55
a/R = 0.032
$V_z \cdot (2\pi a/\Gamma)$ = 0.51
Re = 10830

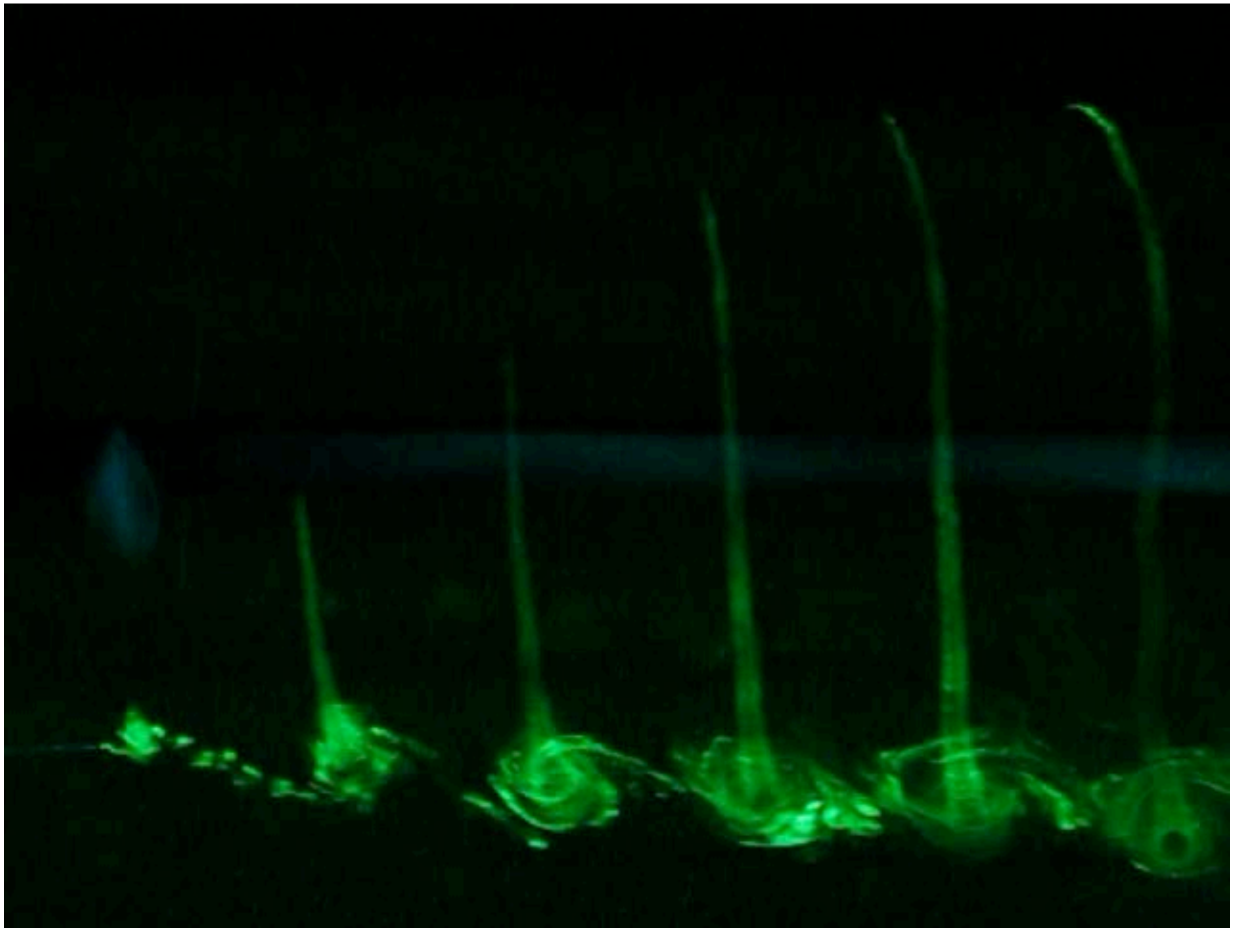
Perturbation vorticity



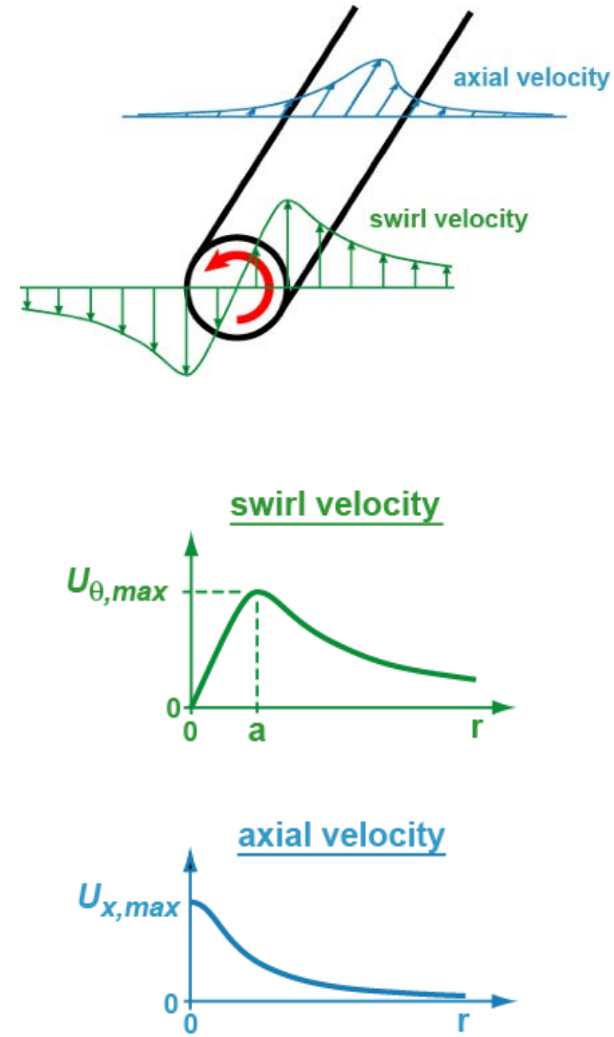
modes
m_1=1, m_2=0



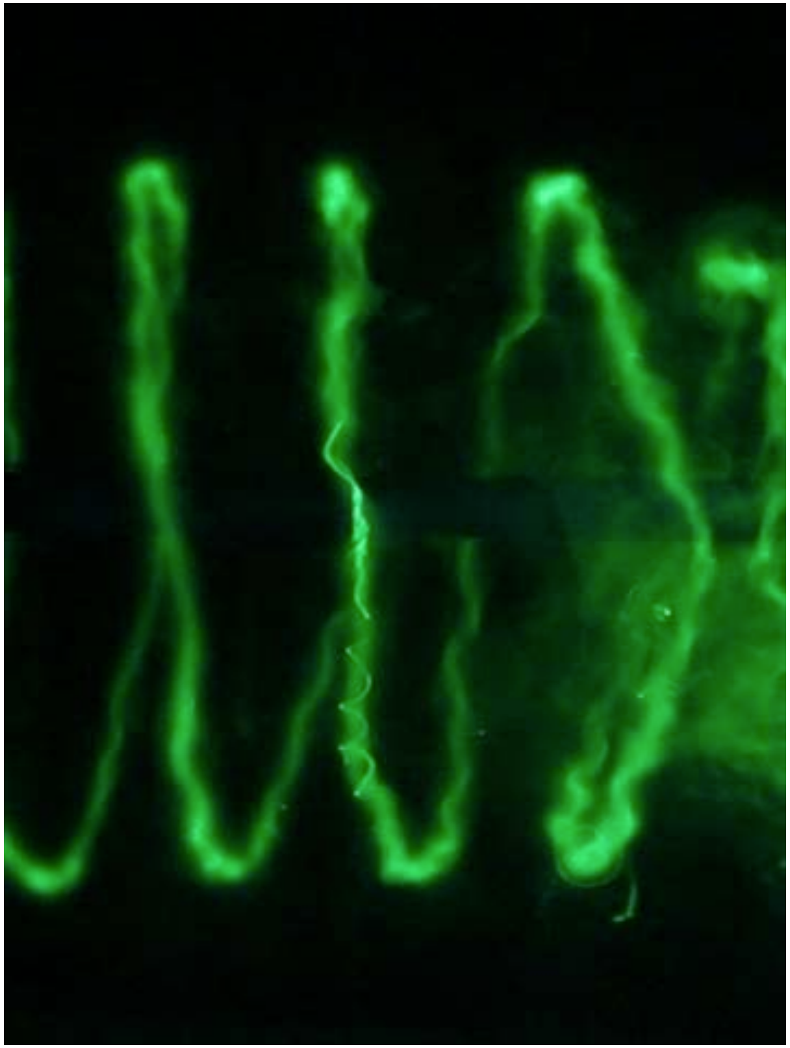
Axial flow in vortex cores



slow motion: real time / 4



Short-wave instability
Experimental observations



slow motion: real time / 8

- **wavelength**

Experiment: $\lambda \approx 6$ a

Theory: $\lambda = 5.7$ a

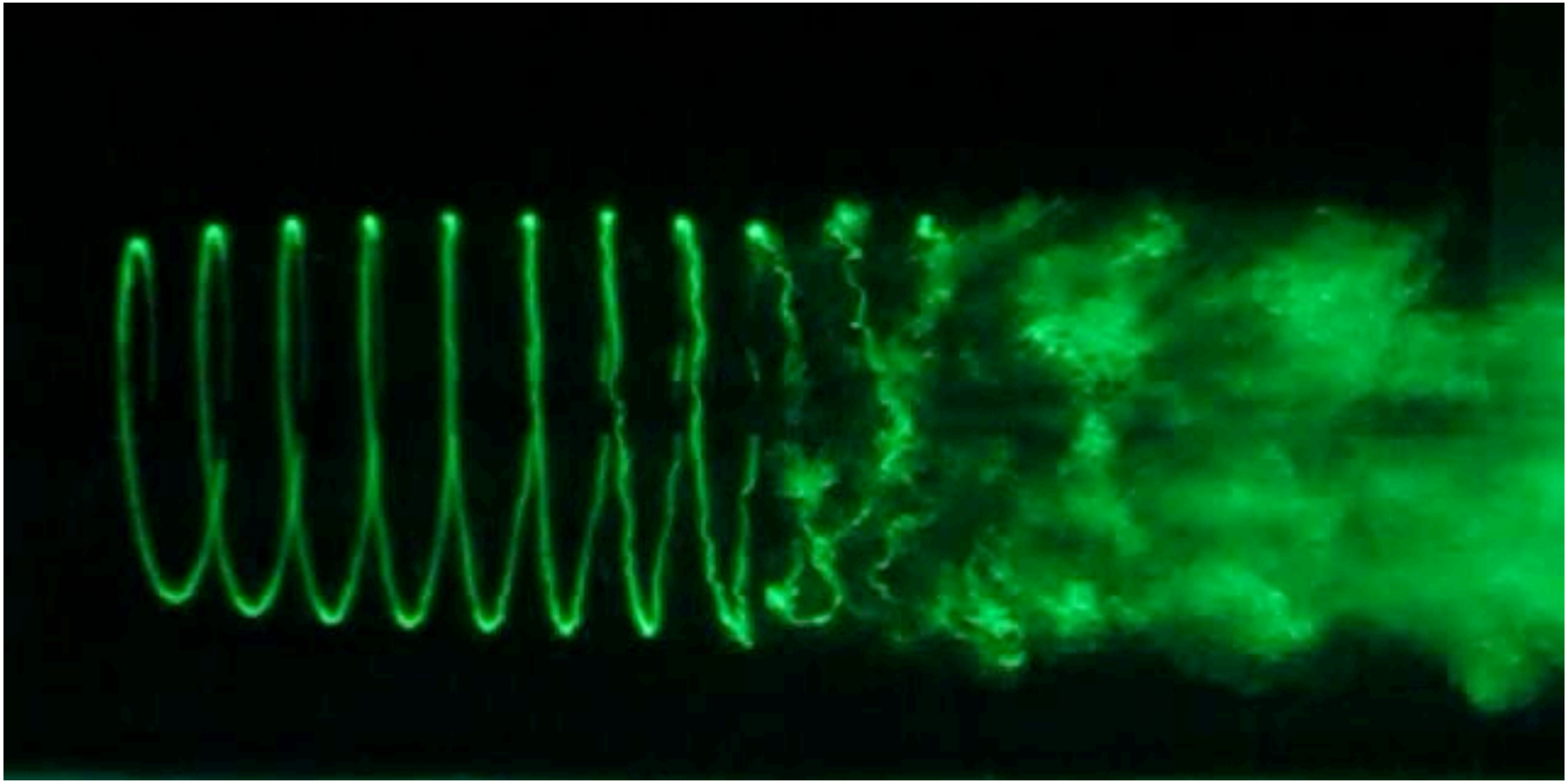
- **mode shape**

Visualisation compatible with curvature instability

- **growth rate**

$\sigma^* = 0.15$

Short-wave instability
Experimental observations



slow motion: real time / 4

Takeaways

➡ **Long-wave instability:** vortex displacement, wavelength $\gg$ core size

➡ **Short-wave instability:** core deformations, wavelength ≈ core size

➡ **Vortex pairs**

- Long-wave Crow instability
  - Strain + auto-rotation + mutual induction
  - Counter-rotating pair → unstable; co-rotating pair → stable
- Short-wave elliptic instability

➡ **Helical vortices**

- Long-wave instability
  - Pairing of neighbouring helix loops
- Short-wave curvature instability
  - Requires axial core flow

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End of Lecture 4